Girls' Angle Bulletin

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To Foster and Nurture Girls' Interest in Mathematics

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From the Founder

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Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

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On the cover: Needles and Noodles, by Claude (Anthropic). For more, see page 24.

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An Interview with Rosa Orellana, Part 1

Rosa Orellana is Professor of Mathematics at Dartmouth College. She obtained her bachelor's degree from California State University, Los Angeles, the first in her family to obtain a college degree. She then earned her doctoral degree in mathematics from the University of California, San Diego under the supervision of Hans Wenzl.

Ken: How did you first become interested in mathematics?

Rosa: It started when I was in seventh grade, though I should tell you that I almost didn't pass fifth grade math. Because of the way it was taught at my school, we were learning long division and it was very difficult to remember. We were given two numbers and I always got confused about which number went where. Finally, I memorized a specific example. Every time I had to do long division, I would refer back to that example to remember where to place the numbers because I had somehow missed the original explanation.

In seventh grade, I took algebra for the first time. For me, it has always been about algebra. We had problems where you have a farm of a certain width and height, and you have to find the area by setting up an equation and solving for x. I remember my teacher used to give us just the final answers to all the problems. I was working on one specific problem; I knew how to do all the other ones, but I couldn't figure this one out. I spent so much time on it. My mother had a rule: Lights out at 10. But that night, I remember going to the window to continue working on it by moonlight. I ended up falling asleep there instead of in my bed. The next day, the first thing I did was run to my teacher. I don't remember many teachers' names, but I still remember hers: Mrs. Robles. I ran to her and said, "I just don't know what I'm doing wrong." In about 20 seconds, she solved the problem. At that moment, I developed a sort of hero worship. I thought, "Wow, I want to be like her and solve problems that fast." Since then, I felt I wanted to be a math teacher. My admiration for her was so great after struggling so hard that it was the first time I thought about teaching math. I wanted to teach that exact course because I wanted to be just like her.

In second grade, when I was learning to add numbers, the husband of one of my sisters told me that I should be a mathematician. I was actually getting the answers wrong, but he told me that the way I went about solving them was very creative. I don't know how he saw that, but he was very kind to me.

Ken: Wow, that's amazing. What school was this?

Rosa: It was a Catholic school in El Salvador, Santa Lucia. But it doesn't exist anymore. When I visited years later when I was in college, the school was gone.

Ken: Do you remember the problem that kept you up at night?

Rosa: No, I just remember it was one of those problems where you have to solve for x. I don't remember the exact problem, just that I was doing something wrong and couldn't figure out what. What happens when you do something wrong in math is that if you firmly believe your approach is

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right, you won't change it. You have to question everything in math. If you believe your initial setup is correct, but it actually isn't, you will never get the right answer because you will never think to change it.

Ken: Did you have similar experiences where you became obsessed with a math problem later during your K12 phase?

Rosa: Yes, it happened again in high school. This was in Los Angeles during the summer after either 9th or 10th grade. I was taking summer school to try to get a head start.

I was taking summer geometry and I completely misunderstood the terminology. One thing that happens in the classroom is that a lot of concepts are assumed to be common knowledge. For instance, dimension is assumed to be understood, but you cannot actually give it a formal definition until you take linear algebra and learn that it is the number of elements in a basis. You just have to accept it until then.

One day, the teacher walked into the classroom, drew a rectangle on the board, and said, "That's a plane." Because he drew a rectangle to represent it, my understanding was that "plane" was just another name for a rectangle. Then, on some of the quizzes, I was asked about all types of configurations involving planes—such as whether two planes can meet at a single point. I thought, "Why not? You have one rectangle touch the corner of another, so planes definitely can meet at a point." I had all these misunderstandings because the teacher assumed we already knew what these things were, so I was making up definitions based on what I assumed he meant.
The teacher, Mr. Hanson, was really amazing, but I got a D on that first quiz because of all those wrong definitions. Even though I was struggling in the course, I never felt like I couldn't learn it. I always felt that I just had to work harder to figure it out. I never thought, "I'm not good at this."

Then we reached the proofs section of the course. Proofs came very naturally to me because we were given all the axioms and definitions to start from right at the top. There were no hidden assumptions we had to make. With the earlier questions, I didn't know what a plane actually was, so I was building configurations with rectangles that you simply cannot do with geometric planes. But when the proofs came, everything was explicitly defined. I had no issues after that and went from getting a D to getting A's on the quizzes.

At the end of the summer term, Mr. Hanson wrote me a personal letter. He wrote that someone with a talent like mine should continue studying mathematics. Unfortunately, I lost the box where I kept that letter, which is a shame because sometimes I feel like I want to read it again. He was so kind. He didn't know that I was already planning to study math, even during the moments when I was not getting an A.

Ken: What an amazing story! It's so ironic, because I think what you did on that test you got a D on actually shows a lot of math talent. It is the kind of example where the grade does not reflect your ability at all.

Rosa: Yeah, and, actually, this kind of misunderstanding happens all the time. Often it results from the student arriving with some intuition about how something *should* be, and they assume a definition that does not correspond to the definition used in mathematics. This can result in incorrect homework, but while grading it, the teacher might not realize there is this type of misunderstanding. This past term, I had 100 students in my class. When you're grading,

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Thank you and best wishes,
Ken Fan
President and Founder
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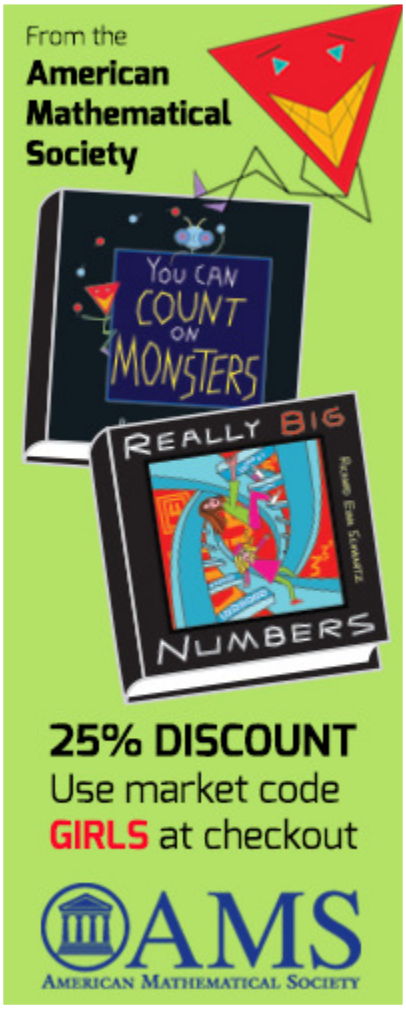

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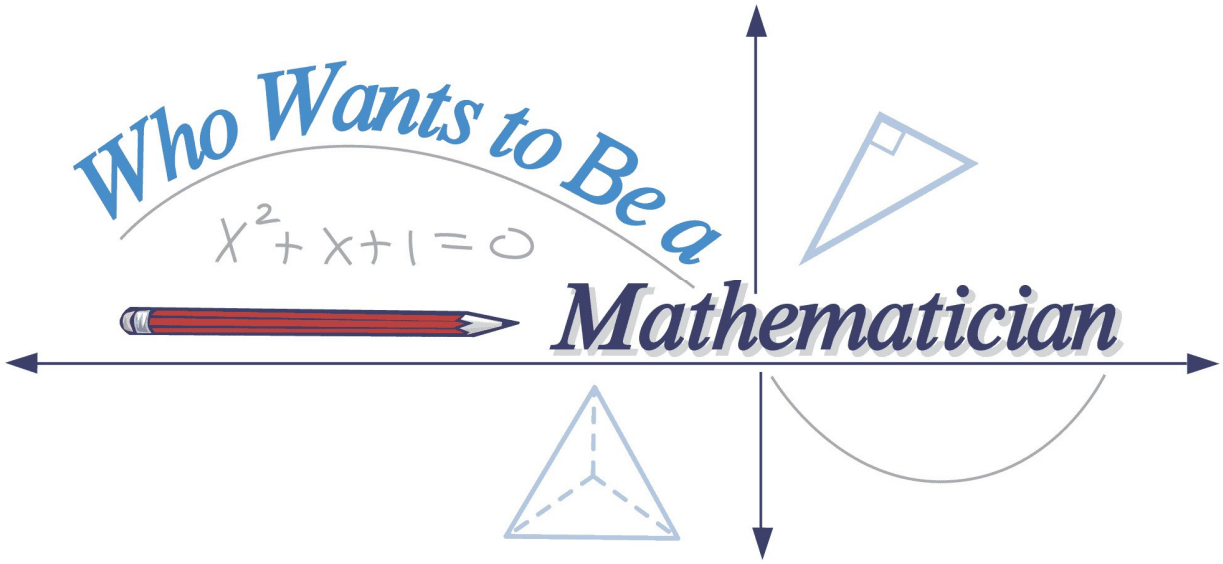

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Taylor Series for Tangent, Part 6

by Ken Fan | edited by Jennifer Sidney

We pick up with Emily and Jasmine staring at the table below of possible ways to partition the permutations in S_3 into two parts, one of size two, the other of size four.

Emily: Perhaps the recursion relation will provide a clue about which partition to take. The recursion formula is

$$c_{k,\,n+1} = \frac{n-k+2}{2}\,c_{k-1,\,n} + 2(k+1)c_{k+1,\,n}$$

(where $c_{-1,\,n}$ is defined to be 0). When $n = 2$, the coefficients 1 and 2 in $p_3(x) = x^2 + 2$ arise as

$$c_{0,\,3} = 2c_{1,\,2}$$

and

$$c_{2,\,3} = c_{1,\,2}.$$

The nth derivative of $\tan(x)$, with respect to x, is

$$\frac{p_n(2\sin(x))}{\cos^{n+1}(x)},$$

where $p_n(x)$ is defined recursively by $p_1(x) = 1$ and

$$p_{n+1}(x) = \frac{(n+1)xp_n(x) + (4-x^2)p'_n(x)}{2},$$

for $n \geq 1$, where $p'_n(x)$ denotes the derivative of $p_n(x)$ with respect to x. Let $c_{k,\,n}$ be the coefficient of x^k in $p_n(x)$. We also define $c_{-1,\,n}$ to be 0. We have

$$c_{k,\,n+1} = \frac{n-k+2}{2}\,c_{k-1,\,n} + 2(k+1)c_{k+1,\,n}.$$

Id.	Part (size 2)	Part (size 4)
1	123 132	213 231 312 321
2	123 213	132 231 312 321
3	123 231	132 213 312 321
4	123 312	132 213 231 321
5	123 321	132 213 231 312
6	132 213	123 231 312 321
7	132 231	123 213 312 321
8	132 312	123 213 231 321
9	132 321	123 213 231 312
10	213 231	123 132 312 321
11	213 312	123 132 231 321
12	213 321	123 132 231 312
13	231 312	123 132 213 321
14	231 321	123 132 213 312
15	312 321	123 132 213 231

Potential ways to partition S_3.

Jasmine: Perhaps this is telling us that each part of the partition should involve permutations that, when written in one-line notation, have as many 1s appearing before 2 as there are permutations where 2 appears before 1. That would leave us with these possibilities. (See the table at the top of the next page.)

Emily: So for the permutations 12 and 21, we need to add in a 3 somewhere, and for each one, we only have three choices. Either the 3 goes to the left, in the middle, or to the right. Somehow, left and right feel symmetric, whereas middle feels different; and only the partition tagged #7 has both permutations in the part of size 2 obtained from 12 and 21 by putting a 3 in the middle. It all sounds a bit arbitrary, but I don't see what else to go by.

Jasmine: There's sense in it. You're saying that putting a 3 at the beginning feels similar to putting it at the end, but different from putting it in the middle. So, for example, between the partitions #2 and #15, neither asserts itself as the one to take, because why should placing 3s at

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the end be favored compared to placing 3s at the beginning (or vice versa)? The only partition that evades this issue is #7. So let's try that. If some compelling reason arises to choose a different one, we can always switch later – assuming that our partition idea is right and that for S_3, the permutations partition as {132, 231} and {123, 213, 312, 321}. Do you see a way to characterize these two parts?

Id.	Part (size 2)	Part (size 4)
2	123 213	132 231 312 321
3	123 231	132 213 312 321
5	123 321	132 213 231 312
6	132 213	123 231 312 321
7	132 231	123 213 312 321
9	132 321	123 213 231 312
11	213 312	123 132 231 321
13	231 312	123 132 213 321
15	312 321	123 132 213 231

Partitions where each part contains as many permutations with 1 coming before 2 as there are with 2 coming before 1.

Emily: The highest number is either in the middle or it's not. But this can't be the right characterization because it implies splitting every S_n into just two parts, which the polynomials tell us would not be the case.

Jasmine: I can't think of anything else … let's see if we can partition S_4. Since $p_4(x) = x^3 + 8x$, our partition should still have two parts, one of size $2^3 = 8$, the other of size $8(2) = 16$. And the recurrence relation tells us they arise as

$$c_{1,4} = 2c_{0,3} + 4c_{2,3}$$

and

$$c_{3,4} = c_{2,3}.$$

This suggests that the part of size 8 is buildable solely from {123, 213, 312, 321}, whereas the part of size 16 will be built from both {123, 213, 312, 321} and {132, 231}.

Emily: We do get eight permutations by sticking a 4 in one of the two middle slots in all possible ways in each of those four permutations: 1423, 1243, 2413, 2143, 3412, 3142, 3421, and 3241.

Jasmine: If we try to continue this pattern, then for S_5, we'd get a part with $8(3) = 24$ permutations, since that's how many permutations we'd get if we inserted 5 "in the middle" in every possible way into those eight permutations.

Emily: Since $p_5(x) = x^4 + 22x^2 + 16$, that would be inconsistent. It's not compatible with the sizes suggested by your idea, which would be 16, 88, and 16.

Jasmine: Wait a sec. The set {123, 213, 312, 321} is obtained by putting 3 at the ends in every possible way, not the middle … so maybe we should try sticking 4 at the ends in every possible way. That would double the number of permutations, and this doubling would continue because going from n to $n + 1$, there are still only two ends: either $n + 1$ is put at the beginning, or it is put at the end. And that sounds like it could correspond to the highest degree term of our polynomials $p_n(x)$, since those are x^{n-1}, which – if our partitioning idea is good – would correspond to a part with 2^{n-1} permutations!

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Emily: That sounds promising … so then we'd get these eight permutations:

4123, 1234, 4213, 2134, 4312, 3124, 4321, and 3214.

Jasmine: Yet these aren't all the permutations that start with 4 or end with 4; there are 12 of those. The missing ones are 4132, 4231, 1324, and 2314. What property separates these 12 permutations that either begin or end with 4 into two different sets: one that hopefully forms the part of size 2^3 in our partition, and the other four in the other part with 16 elements?

Emily: Reading them through, those 8 that we hope will form a single part … they all have a "down-up" character; that is, the numbers start by going down, then they ascend. For example, 2134 starts at 2, goes down to 1, then up to 3, then 4. The missing ones, by contrast, go "down-up-down" or "up-down-up."

Jasmine: Neat! Hold on. What about the permutation 1234? That just goes up. And 4321 only goes down. I suppose we could take the permutations that go down or up, then possibly switch direction one more time, but no more than one more time.

Emily: But none of them go up then down. They only go down then up, or just down, or just up.

Jasmine: I see. Okay, what about the twelve permutations that don't begin or end with 4? Those should join 4132, 4231, 1324, and 2314 to form the other part.

Emily: Here are the permutations that don't begin with 4 or end with 4:

1243 1342 1423 1432 2143 2341 2413 2431 3142 3241 3412 3421

Let's indicate the up/down pattern using a sequence of U's and D's, for Up and Down. So these go UD, UD, UDU, UD, DUD, UD, UDU, UD, DUD, DUD, UDU, and UD, respectively, and the four permutations they join go DUD, DUD, UDU, and UDU.
Jasmine: And it looks like of those 16 elements, eight come from the $c_{0,3}$ partition {132, 231} and eight come from the $c_{2,3}$ partition {312, 321, 123, 213}.

Emily: Maybe defining parts of the partition by specifying which up/down patterns occur in permutations works, but it isn't very clean; and it's not clear at all how the various up/down patterns should be assigned. I wish there were a better way to describe which part of a partition each permutation belongs to.

Jasmine: Perhaps if we try to partition S_5, we'll get some clarity on this.

Emily: Maybe, but I feel like I'm already on shaky ground. Well, I guess the choice we made for the partition of S_3 does seem to be curiously singled out. The partition for S_4 is less convincing to me, but at least the permutations with a fixed up/down pattern all fall into the same part together. That seems promising. But which up/down patterns go to which parts? Why should DU and UD be in different parts?

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Jasmine: Let's neatly list our proposed partition of S_4:

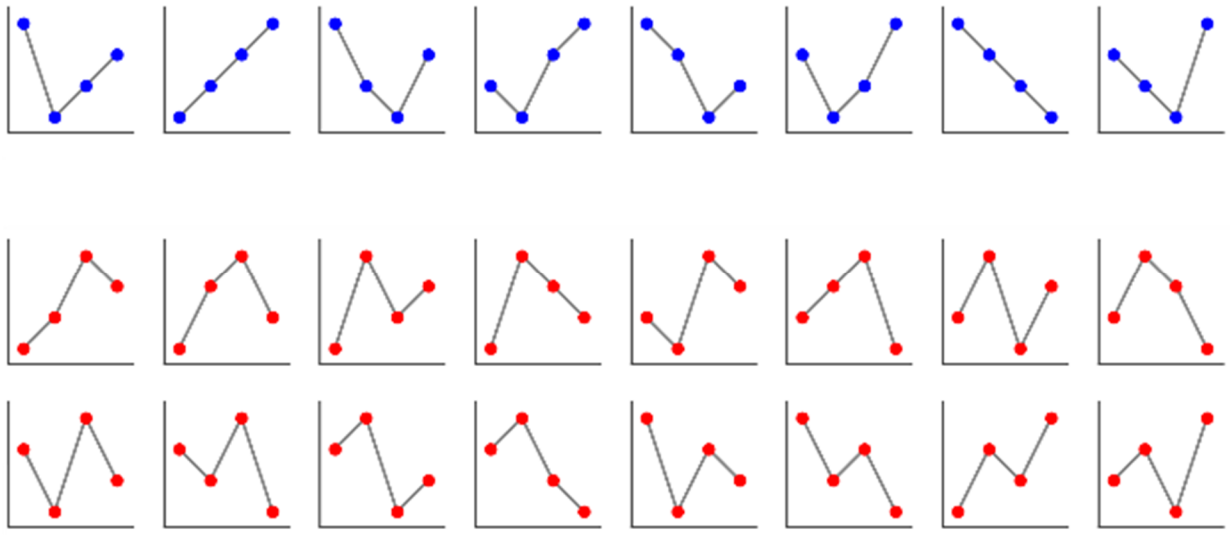
x^3 part: 4123 1234 4213 2134 4312 3124 4321 3214

$8x$ part: 1243 1342 1423 1432 2143 2341 2413 2431 3142 3241 3412 3421
4132 4231 1324 2314

Emily: Since we're thinking in terms of ups and downs, I feel like drawing graphs of these permutations.

Jasmine: Let's do that!

Emily and Jasmine write a Python program to produce this image. The x^3 part is in blue.



Emily: Hey, something does jump out from these pictures!

Jasmine: The permutations in the $8x$ part are exactly the permutations that have "internal peaks"?

Emily: Exactly!

Jasmine: For a permutation p, let's define an internal peak as a position i, such that $1 < i < n$, and

$$p(i-1) < p(i) \text{ and } p(i) > p(i+1).$$

Emily: With that definition, we can say that the x^3 part is the set of permutations with zero internal peaks and the $8x$ part is the set of permutations with one internal peak.

Jasmine: That's clean!

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Emily: If we can define parts by the number of internal peaks, since $p_5(x) = x^4 + 22x^2 + 16$, in S_5, we should find $2^4 = 16$ permutations with zero internal peaks, $22(2^2) = 88$ with one internal peak, and 16 with two internal peaks, and none with more than two internal peaks. Is this true?

Jasmine: Well, there definitely aren't permutations in S_5 with more than two internal peaks, because there are only three internal positions and they can't all be peaks. In fact, internal peaks can never be adjacent.

Emily: True. Let's see if there are 16 with no internal peaks. The permutations that are U, D, or DU all have this property, and we know there are 16 of those, since those are all obtained from the eight permutations in x^3 for S_4 by placing a 5 at the beginning or end of the permutation. It's also true that 5 cannot appear at an internal position, because 5 is bigger than every other number and would then be an internal peak. If the 5 is erased, the result can't have an internal peak, because that internal peak would persist after a 5 is placed at either end. So there are exactly 16 permutations in S_5 with no internal peaks!

Jasmine: Nice. How about permutations with two internal peaks? Are there 16? Since internal peaks cannot be next to each other, a permutation in S_5 with two internal peaks must have them at locations 2 and 4. I think one of these must be sent to 5, because 5 can't be at an end; otherwise, the number next to it, which has to be in position 2 or 4, could not be an internal peak. Nor can 5 be in the middle, since that would make position 3 an internal peak. So let's assume for now that 5 is in position 2. The 4 could go in position 1, but it cannot go in positions 3 or 5, since that would prevent an internal peak at position 4. But the value 4 could form the internal peak at position 4, so the 4 can go either in position 1 or 4. If it's in position 4, then we can place the 1, 2, and 3 in any order in positions 1, 3, and 5. This gives us six permutations. And if the 4 is in position 1, we have no choice but to put the 3 in position 4 and sandwich it by the 1 and 2 in either order. That gives us two more such permutations, for a total of eight. And, by symmetry, there are eight such permutations with the 5 in position 4 … they're just the eight with the 5 in position 2 written in reverse!

Emily: Wow, I can't believe this is working!

Jasmine: Since permutations in S_5 can only have zero, one, or two internal peaks, and we've accounted for all the permutations with zero or two peaks, the number of permutations with one internal peak must be $5! - 16 - 16 = 120 - 16 - 16 = 88$. That's exactly what we were hoping for!

Emily: So we have a working conjecture: The permutations in S_n can be partitioned by counting the number of internal peaks. Furthermore, there should be $c_{n-1-2i,\,n}(2^{n-1-2i})$ permutations with i internal peaks, assuming that $i = 0$ corresponds to the highest degree monomial x^{n-1}, and increases by 1 when going through the terms of the polynomial in decreasing order of degree.

Jasmine: Maybe we can prove this by induction. It's true for $n \leq 5$.

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Emily: Seems like a good plan; we have the recurrence relation for the $c_{k,n}$, so we can derive the recurrence relation we need these partitions to satisfy. That is, if we define $m_{i,n}$ to be the number of permutations in S_n with i internal peaks, we need to prove that

$$\begin{aligned} m_{i,n+1} &= c_{n-2i,n+1}(2^{n-2i}) \\ &= ((i+1)c_{n-1-2i,n} + 2(n+1-2i)c_{n+1-2i,n})2^{n-2i} \\ &= 2(i+1)c_{n-1-2i,n}(2^{n-1-2i}) + (n+1-2i)c_{n-1-2(i-1),n}2^{n-1-2(i-1)} \\ &= 2(i+1)m_{i,n} + (n+1-2i)m_{i-1,n}. \end{aligned}$$

Jasmine: From this recursion relation, it looks like we have to try to build permutations in S_{n+1} that have i internal peaks from permutations in S_n with either i or $i-1$ internal peaks. So suppose we have a permutation p in S_n with i internal peaks. Can we add $n+1$ to its one-line notation so that we don't change the number of internal peaks?

Emily: We can always put $n+1$ at the beginning or end. That won't introduce a new peak because the first and last positions are not internal, and the value next to the $n+1$ won't suddenly become a new peak since it is sitting right next to $n+1$, the highest value around. That gives us two permutations for each p, but the recursion relation suggests that we need to build $2(i+1)$ permutations for each p.

Jasmine: So there would have to be ways to put the $n+1$ into an internal position without changing the number of internal peaks. But no matter where $n+1$ is placed internally, it will become an internal peak because it's the largest number, sandwiched by smaller numbers.

Emily: That means that one of the internal peaks of p would have to lose its status. I see! We place $n+1$ *next to* an internal peak! The one it is placed next to loses its status as an internal peak because it is now next to $n+1$. But $n+1$ takes over in that role, preserving the number of internal peaks.

Jasmine: Beautiful! And the old internal peak is the only one that is lost, because $n+1$ cannot be next to two internal peaks because that would imply that p has adjacent internal peaks, which is impossible. And if $n+1$ is not placed next to an internal peak but is placed somewhere other than at the ends, then all the internal peaks of p will carry over *and* we'll get another internal peak in $n+1$ itself, increasing the number of internal peaks by one. So for each internal peak, we get two ways to build a permutation in S_{n+1} with i internal peaks. That's $2i$ permutations, plus the two we get by putting $n+1$ at the ends, and there's our $2(i+1)$!

Emily: Perfect, and that suggests to us how to build permutations in S_{n+1} with i internal peaks from a permutation q in S_n with $i-1$ internal peaks. To do that, we now *need* to introduce a new internal peak. So we must insert $n+1$ in between two numbers, neither of which is an internal peak of q, which means we cannot place it at an end. Any spot that meets these criteria will work. So the question is: How many such spots are there?

Jasmine: Since we know exactly how many spots *do not* increase the number of internal peaks, we can just subtract that from the total number of places $n+1$ can be placed. The $n+1$ can be placed to the left of any number, or at the very end. So there are $n+1$ places where we can add

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the $n + 1$ to the one-line notation. When there are $i - 1$ internal peaks, the number of places that do not increase the number of internal peaks is $2(i - 1)$ for the places next to an internal peak, plus two for each end, giving us a total of $2i$ places that don't work. Therefore, there are $n + 1 - 2i$ places where we could insert $n + 1$ and *increase* the number of internal peaks, and that's exactly what the recursion formula wants!

Emily: I can't believe how cleanly this all worked out! We are, in fact, also creating every single permutation in S_{n+1}, because we're taking every permutation in S_n and inserting the $n + 1$ in each of the $n + 1$ possible places in the one-line notation. We're just tacking on the internal peak tracking to see that everything organizes compatibly with the recurrence relation. Your hunch was true! Congratulations!

Jasmine: This is so cool. We set out to find the Taylor expansion of the tangent function, and ended up finding a family of polynomials whose terms, $c_{k,n}x^k$, count – after substituting 2 for x – the number of permutations in S_n that have $i = (n - 1 - k)/2$ internal peaks!

Emily: We just have to take note that k and n must have opposite parity in that formula, which is fine because when k and n have the same parity, we know that $c_{k,n} = 0$.

Jasmine: Good point!

Emily: But wait! It's the coefficients $c_{0,n}$ for odd n that, after division by $n!$, give us the coefficients in the Taylor expansion of $\tan(x)$. That means that for odd n, the coefficient of $x^n/n!$ in the Taylor expansion of $\tan(x)$ *is* the number of permutations p in S_n that have the maximal number, $(n - 1)/2$, of internal peaks.

Jasmine: You're right! And since internal peaks cannot be adjacent to each other, the only way to get $(n - 1)/2$ internal peaks is for every even-numbered place in the one-line notation to be an internal peak. That means that the one-line notation must be a maximal zigzag! We must have

$$p(1) < p(2) > p(3) < p(4) > p(5) < p(6) > \ldots < p(n-3) > p(n-2) < p(n-1) > p(n).$$

Emily: Incredible! We found a combinatorial interpretation of the coefficients in the Taylor expansion of the tangent function!

To be continued …

The connection between the coefficients in the Taylor series of $\tan(x)$ and zigzag permutations was first discovered by Désiré André in 1879–1881; the more complete peak-by-peak refinement is a later, related development. *-Editor*

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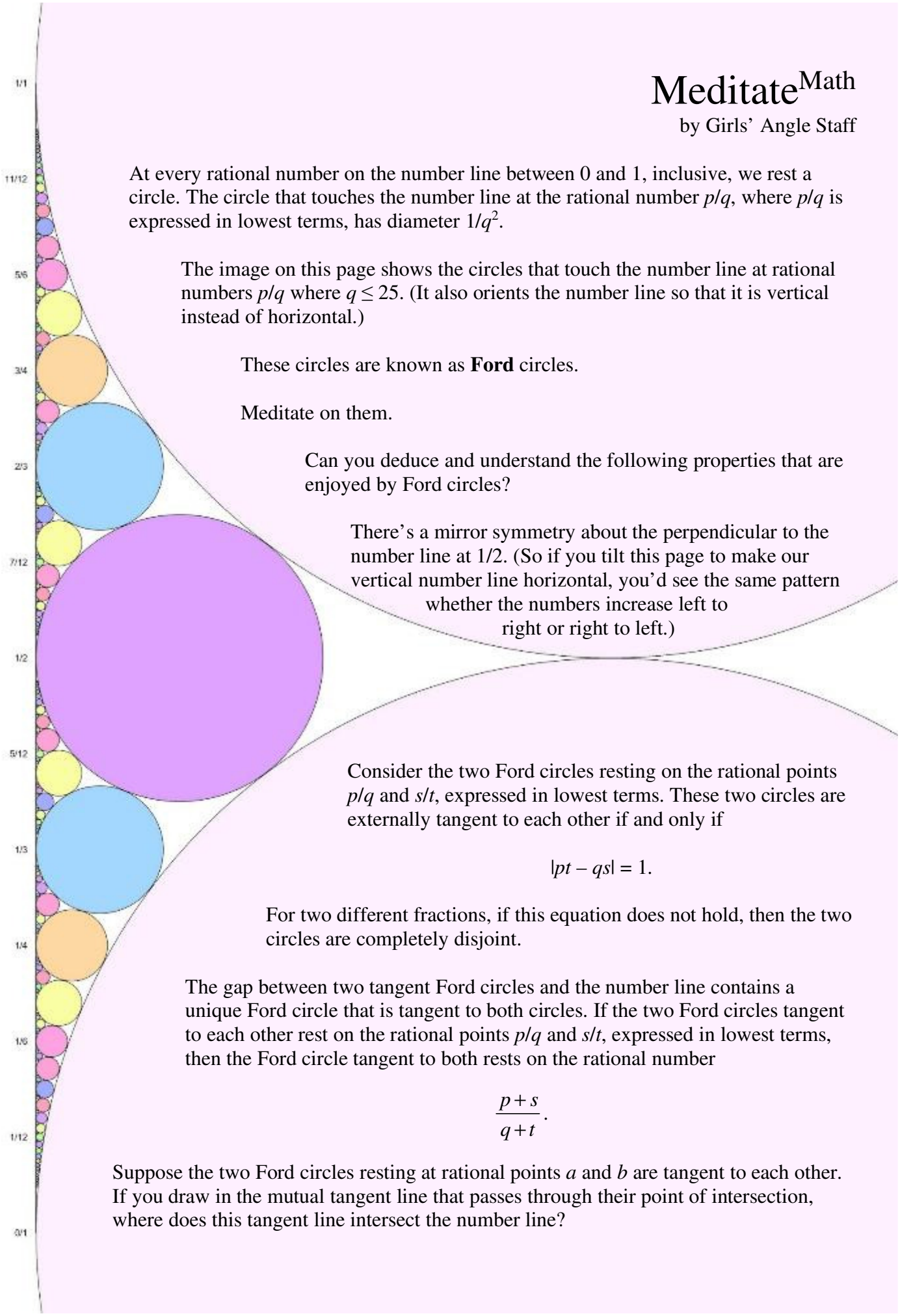

Meditate^Math

by Girls' Angle Staff

At every rational number on the number line between 0 and 1, inclusive, we rest a circle. The circle that touches the number line at the rational number p/q, where p/q is expressed in lowest terms, has diameter $1/q^2$.

The image on this page shows the circles that touch the number line at rational numbers p/q where $q \leq 25$. (It also orients the number line so that it is vertical instead of horizontal.)

These circles are known as **Ford** circles.

Meditate on them.

Can you deduce and understand the following properties that are enjoyed by Ford circles?

There's a mirror symmetry about the perpendicular to the number line at 1/2. (So if you tilt this page to make our vertical number line horizontal, you'd see the same pattern whether the numbers increase left to right or right to left.)

Consider the two Ford circles resting on the rational points p/q and s/t, expressed in lowest terms. These two circles are externally tangent to each other if and only if

$$|pt - qs| = 1.$$

For two different fractions, if this equation does not hold, then the two circles are completely disjoint.

The gap between two tangent Ford circles and the number line contains a unique Ford circle that is tangent to both circles. If the two Ford circles tangent to each other rest on the rational points p/q and s/t, expressed in lowest terms, then the Ford circle tangent to both rests on the rational number

$$\frac{p+s}{q+t}.$$

Suppose the two Ford circles resting at rational points a and b are tangent to each other. If you draw in the mutual tangent line that passes through their point of intersection, where does this tangent line intersect the number line?

Positivity of Harned–Rittenberg Polynomials

by Ken Fan and Claude (Anthropic)

In *Counting NIM, Part 2*,[1] Milena Harned and Miriam Rittenberg asked and answered: How many ways are there to entirely remove plates from two stacks of plates if you're allowed to remove any number of plates off the top of a single stack on each turn? For example, here are the 14 ways the plates can be removed from two stacks of two plates each.

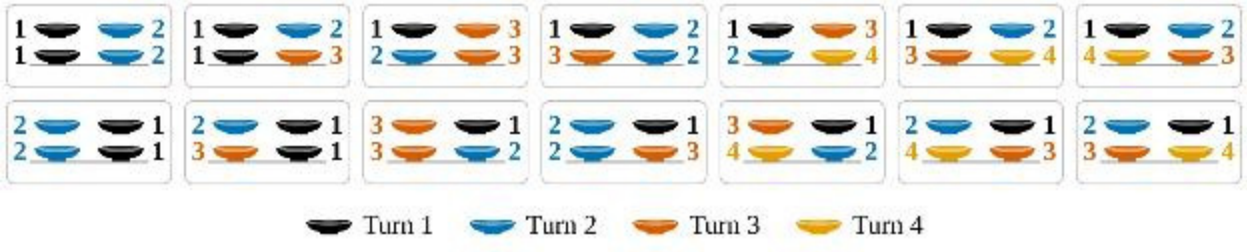

Figure 1. The 14 ways to remove 4 plates arranged into 2 stacks of 2 plates each. Image by Claude (Anthropic).

Harned and Rittenberg showed that if one stack has n plates and the other has m plates, then the number of ways to remove them is

$$\frac{2^{n-m-1}}{m!}P_m(n)\,,$$

where $P_m(n)$ is a monic polynomial of degree m with integer coefficients. Here are the first few polynomials $P_m(n)$:

m	$P_m(n)$
0	1
1	$3 + n$
2	$26 + 13n + n^2$
3	$330 + 215n + 30n^2 + n^3$
4	$5496 + 4350n + 851n^2 + 54n^3 + n^4$
5	$113160 + 104314n + 25835n^2 + 2365n^3 + 85n^4 + n^5$

This family of polynomials appears to have been discovered by Harned and Rittenberg; their triangle of coefficients is sequence A289329 on the On-Line Encyclopedia of Integer Sequences (OEIS), which was added by our Head Mentor at the time, Jane Wang.

Note that the number of ways to remove the plates is the same if the two stacks are swapped. This means that the polynomials $P_m(n)$ satisfy the functional equation

$$4^n n!P_m(n) = 4^m m!P_n(m).$$

[1] See pages 6-10 of Volume 10, Number 5 of the *Girls' Angle Bulletin*.

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Harned and Rittenberg also conjectured that all the coefficients of $P_m(n)$ are positive.

Since they defined their polynomials, various people have tried to prove their conjecture, but it resisted attempts. Recently, I mentioned their OEIS entry to Anthropic's Claude. Without even asking it to prove the positivity conjecture, Claude went ahead and provided a slick proof, and since I told both Milena and Miriam that I'd put a proof of that conjecture in this *Bulletin* as soon as I heard of one, I present Claude's proof here. It's a nifty proof that uses a change of variables in a generating function.

Here's Anthropic's Claude, lightly edited:

Let $f(n, m)$ be the number of different ways plates can be removed, so

$$f(n, m) = \frac{2^{n-m-1}}{m!} P_m(n) .$$

The proof has four steps. Step 1 is Harned and Rittenberg's counting formula. Step 2 packs the numbers $f(n, m)$ into a generating function. Step 3 rewrites that function in a new variable, which gives a formula for $P_m(n)$ with no subtractions in it. Step 4 reads off positivity. Since $P_0(n) = 1$, there is nothing to prove for $m = 0$, so from now on $m \geq 1$.

Step 1. Counting the games.

Harned and Rittenberg counted the games directly. Suppose that in some game the n-stack is emptied in i moves and the m-stack in j moves. Three independent choices then determine the game:

1. How the n-stack is split into i removals, from the top down. This is the same as placing $i - 1$ dividers in the $n - 1$ gaps between plates, so there are $\binom{n-1}{i-1}$ choices.

2. How the m-stack is split into j removals: $\binom{m-1}{j-1}$ choices.

3. The order of the $i + j$ moves. Once you know which of the $i + j$ moves are on the first stack, the game is determined, so there are $\binom{i+j}{i}$ choices.

Adding over all possible i and j gives their formula, valid for $n \geq 1$:

$$f(n,m) = \sum_{i=1}^{n} \sum_{j=1}^{m} \binom{n-1}{i-1} \binom{m-1}{j-1} \binom{i+j}{i}.$$

Let's check to see that this formula gives us the 14 ways shown in Figure 1, where $n = m = 2$. The sum is ${}_2C_1 + {}_3C_1 + {}_3C_2 + {}_4C_2 = 2 + 3 + 3 + 6 = 14$.

This formula is exact, but it is a big sum, and nothing about it shows that a polynomial is hiding inside. Generating functions bring the polynomial out.

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Step 2. The generating function.

Fix $m \geq 1$. Instead of studying $f(1, m), f(2, m), f(3, m), \ldots$ one at a time, put them all into one power series to create their generating function[2]:

$$F_m(z) = \sum_{n \geq 1} f(n,m) z^n = f(1,m)z + f(2,m)z^2 + f(3,m)z^3 + \ldots .$$

We need two facts. Both come from multiplying geometric series together.

Tool A. For $i \geq 1$,

$$\sum_{n \geq 1} \binom{n-1}{i-1} z^n = \left(\frac{z}{1-z}\right)^i .$$

Why: $\frac{z}{1-z} = z + z^2 + z^3 + \ldots$. When you multiply i copies of this series, each way to get z^n comes from picking $z^{a_1}, \ldots, z^{a_i}$ with $a_1 + \ldots + a_i = n$ and every $a_k \geq 1$. That is exactly a way to split a stack of n plates into i removals, so the coefficient of z^n is $\binom{n-1}{i-1}$.

Tool B. For $j \geq 0$,

$$\sum_{i \geq 0} \binom{i+j}{i} u^i = \frac{1}{(1-u)^{j+1}} .$$

Why: Because $\binom{i+j}{i} = \binom{i+j}{j}$, this generating function is the same as substituting $i = j + 1$ and $z = u$ in Tool A, except that the indexing begins later in Tool A, so after substituting, we have to divide by u^{j+1}.

Putting them together. Multiply the formula from Step 1 by z^n and sum over all $n \geq 1$. Tool A turns the sum over n into powers of $u = \frac{z}{1-z}$:

$$F_m(z) = \sum_{n \geq 1} \sum_{i=1}^{n} \sum_{j=1}^{m} \binom{n-1}{i-1} \binom{m-1}{j-1} \binom{i+j}{i} z^n = \sum_{j=1}^{m} \binom{m-1}{j-1} \sum_{i \geq 1} \binom{i+j}{i} u^i .$$

The inner sum is Tool B's sum without its $i = 0$ term, which is 1. So it equals $\frac{1}{(1-u)^{j+1}} - 1$. Since $1 - u = \frac{1-2z}{1-z}$, set $v = \frac{1}{1-u} = \frac{1-z}{1-2z}$.

[2] For an introduction to generating functions, see Laura Pierson's Summer Fun problem set on pages 25-27 of Volume 11, Number 5 of this *Bulletin*.

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Using the binomial theorem, and the fact that $\sum_{j=1}^{m}\binom{m-1}{j-1}=2^{m-1}$,

$$F_m(z)=\sum_{j=1}^{m}\binom{m-1}{j-1}\left(v^{j+1}-1\right)=v^2(1+v)^{m-1}-2^{m-1}.$$

Finally, $1+v=\frac{2-3z}{1-2z}$, so

$$F_m(z)=\frac{(1-z)^2(2-3z)^{m-1}}{(1-2z)^{m+1}}-2^{m-1}.$$

Check with $m = 1$: $\frac{(1-z)^2}{(1-2z)^2}-1 = 2z + 5z^2 + \ldots$, and indeed $f(1, 1) = 2$ and $f(2, 1) = 5$.

Step 3. A change of variables.

The factor 2^n in $f(n, m)$ is from the denominator $(1 – 2z)^{m+1}$, because $\frac{1}{1-2z} = 1 + 2z + 2^2z^2 + \ldots$.
The key idea is to rewrite everything in terms of the quantity in that denominator $1 – 2z$. That is, let $s = 1 – 2z$, so that

$$1-z=\frac{1+s}{2}\quad\text{and}\quad 2-3z=\frac{1+3s}{2}.$$

Both new expressions contain only plus signs. Substituting them,

$$\frac{(1-z)^2(2-3z)^{m-1}}{(1-2z)^{m+1}}=\frac{(1+s)^2(1+3s)^{m-1}}{2^{m+1}s^{m+1}}.$$

Call the numerator $Q_m(s)$ and multiply it out:

$$Q_m(s)=(1+s)^2(1+3s)^{m-1}=c_0+c_1s+c_2s^2+\cdots+c_{m+1}s^{m+1}.$$

Every c_j is a positive integer, because Q_m is a product of polynomials whose coefficients are all positive integers. Also $c_0 = Q_m(0) = 1$.

Divide by s^{m+1}. The term c_js^j becomes c_j/s^{m+1-j}. Writing $k = m – j$, the terms with $j = 0, 1, \ldots, m$ become $\frac{c_{m-k}}{(1-2z)^{k+1}}$ for $k = 0, 1, \ldots, m$. The one remaining term, $j = m + 1$, is the constant c_{m+1}, which only affects z^0. So for $n \geq 1$,

$$f(n,m)=\frac{1}{2^{m+1}}\sum_{k=0}^{m}c_{m-k}\cdot\left[\text{coefficient of } z^n \text{ in } \frac{1}{(1-2z)^{k+1}}\right].$$

Tool B with $u = 2z$ (and k in place of j) says that coefficient is $\binom{n+k}{k}2^n$, and

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$$\binom{n+k}{k} = \frac{(n+1)(n+2)\cdots(n+k)}{k!}.$$

Therefore, for every $n \geq 1$,

$$f(n,m) = 2^{n-m-1} \sum_{k=0}^{m} \frac{c_{m-k}}{k!}(n+1)(n+2)\cdots(n+k).$$

Compare this with $f(n, m) = \frac{2^{n-m-1}}{m!} P_m(n)$. Both give the same value for every $n \geq 1$, and two polynomials that agree at infinitely many points are identical. So

$$P_m(n) = \sum_{k=0}^{m} \frac{m!}{k!} c_{m-k}\,(n+1)(n+2)\cdots(n+k),$$

where the $k = 0$ term is just $m!c_m$, since an empty product equals 1.

Step 4. Why the coefficients are positive.

Each summand in the formula for $P_m(n)$ is a polynomial in n whose coefficients (up to its own degree) are all positive integers. Furthermore, the term corresponding to $k = m$ is given by $c_0(n + 1)(n + 2)\cdots(n + m)$, which is a polynomial of degree m (whose coefficients are all positive integers). Therefore every coefficient of $P_m(n)$ is a positive integer. □

The same term shows that P_m has degree m and leading coefficient 1, as every other term has degree less than m. So this argument also gives a new proof of Harned and Rittenberg's result.

Example: $m = 2$. Here $Q_2(s) = (1 + s)^2(1 + 3s) = 1 + 5s + 7s^2 + 3s^3$, so $c_0 = 1$, $c_1 = 5$, and $c_2 = 7$. The formula gives

$$P_2(n) = 2 \cdot 7 + 2 \cdot 5(n + 1) + (n + 1)(n + 2) = 26 + 13n + n^2,$$

which matches the list above.

Why the change of variables works.

The formula in Step 1, and the formula on the OEIS page, both naturally express $P_m(n)$ using binomial coefficients. Multiplying those out produces minus signs. The minus signs all cancel in the end, but they hide the positivity.

The change of variables $s = 1 - 2z$ switches to the products $(n + 1)(n + 2)\cdots(n + k)$, which have no minus signs at all. It amounts to expanding $F_m(z)$ around $z = 1/2$, the only point where F_m blows up. Since $\frac{1}{1-az} = \sum a^n z^n$ blows up at $z = 1/a$, a blow-up at $z = 1/2$ is where the factor 2^n in $f(n, m)$ comes from, and because F_m blows up there like $1/(1 - 2z)^{m+1}$, Tool B shows that the factor multiplying 2^n is a polynomial in n of degree m. The proof works because both $1 - z$ and $2 - 3z$ turn into expressions with only plus signs. If either one had picked up a minus sign, this argument wouldn't have shown positivity.

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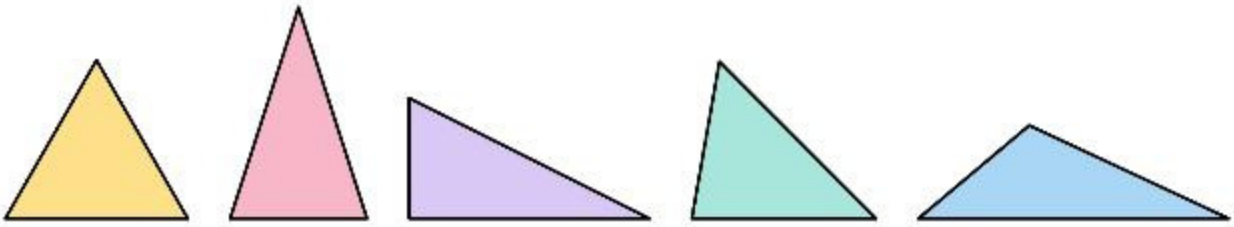
Mathematical Buffet

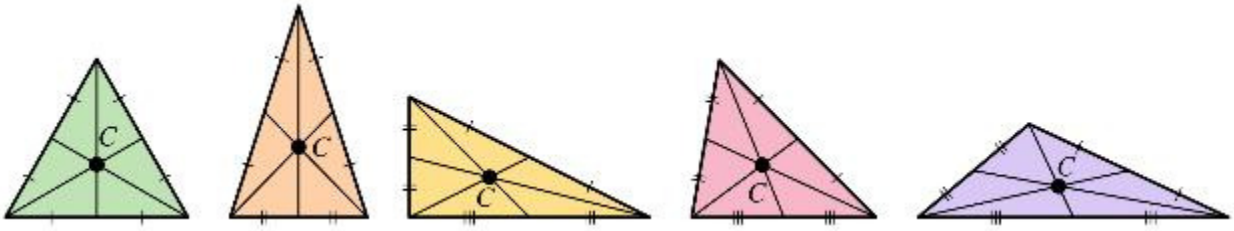
Triangles, Part 2 | By Girls' Angle Staff

The last time we featured triangles in this column, Ashley Wang presented triangles in the real world. In this Mathematical Buffet, we're going to showcase mathematical concepts that are associated with triangles. To illustrate, here are our main actors:



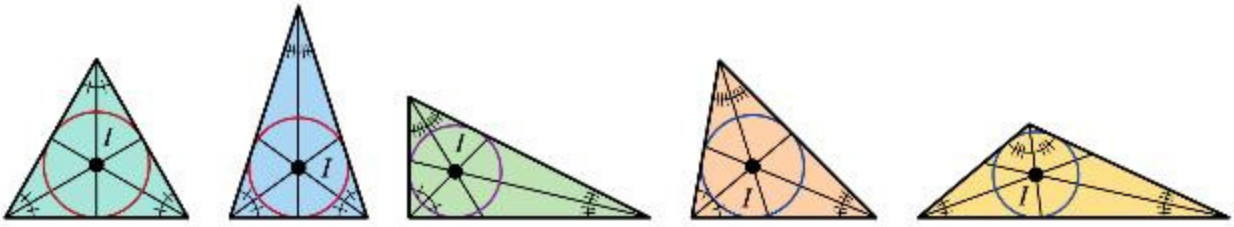
They are, left to right, the equilateral triangle, an isosceles triangle, a right triangle, a scalene acute triangle, and a scalene obtuse triangle.

And the first concept is that of the **median** and the **centroid**:



The medians are the lines that connect a vertex to the midpoint of the opposite side. They intersect in the centroid (*C*), where you should place your finger if you want to balance the triangle on your fingertip.

Next up are **angle bisectors**, the **incenter**, and the **incircle**:



The angle bisectors are the lines that split each corner angle in half. These lines meet at the incenter (*I*), which is the center of the incircle, the circle inside that just touches the three sides.

After the incircle comes the **circumcircle**, **circumcenter**, and **perpendicular bisectors**. For every triangle, there is a unique circle, the circumcircle, that passes through all three of its vertices. Its center, the circumcenter (*O*), sits where the lines that are perpendicular to the sides and pass through their midpoints have a common intersection. See the top of the next page.

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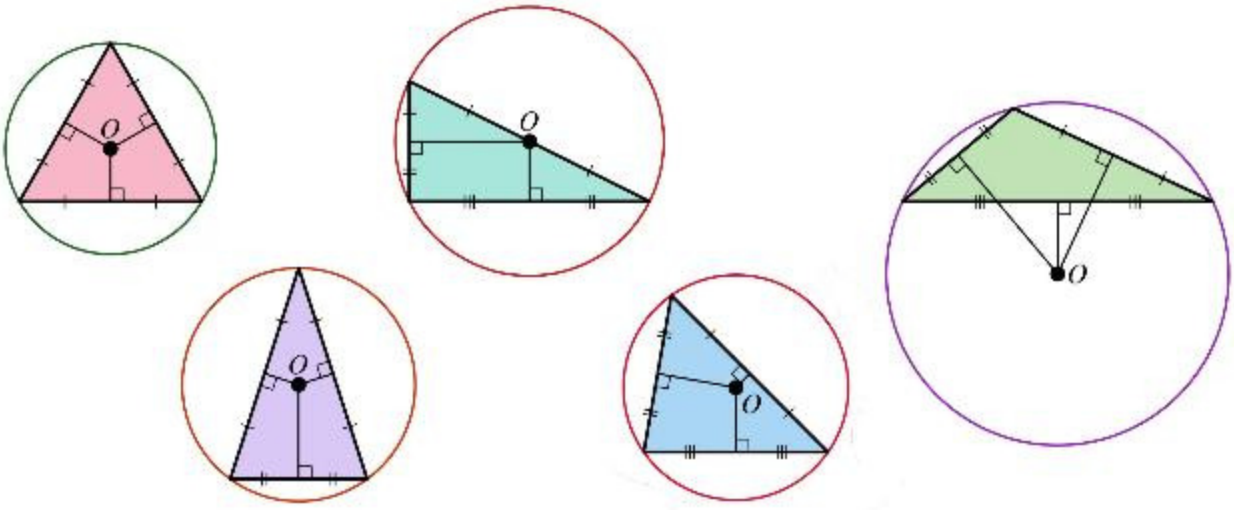
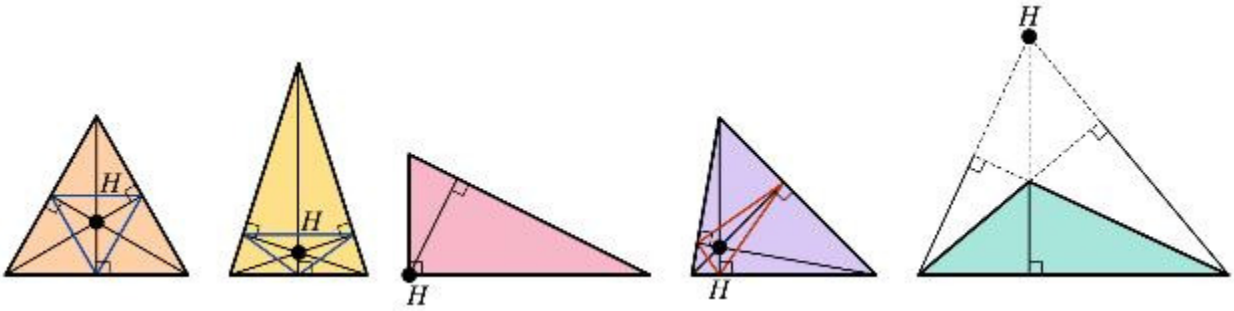
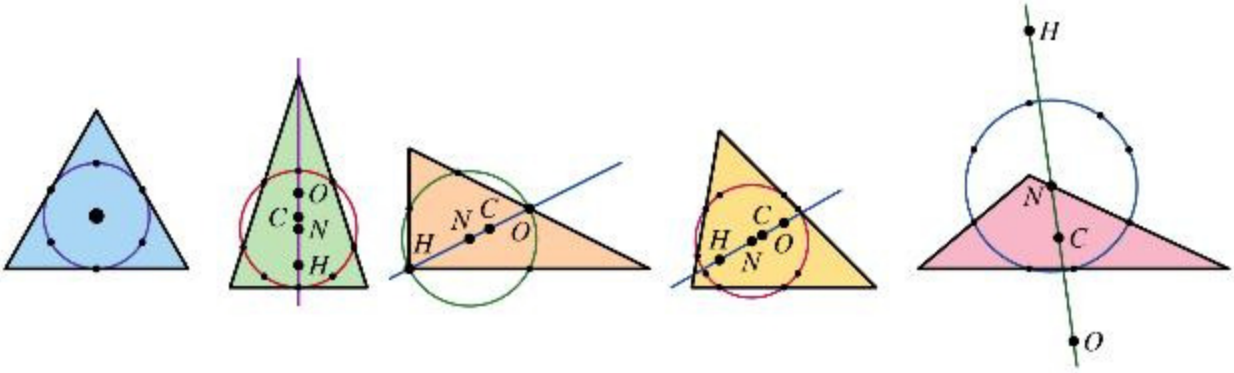


If you've ever computed the area of a triangle using the formula "one-half base times height," you've probably drawn an **altitude**, which is the line through a vertex perpendicular to the opposite side. The three altitudes intersect in the triangle's **orthocenter** (*H*):



In acute triangles, a fun fact about these altitudes is that there is a billiard ball path that bounces where the altitudes meet the sides and loops forever. This loop is also the shortest closed path that touches all three sides, which Giovanni Fagnano showed in 1775.

In every triangle, there's a very special circle called the **9-point circle**. It's the unique circle that passes through the midpoints of the sides. This circle also happens to pass through the feet of the altitudes that we just met, as well as the three points that are halfway from the orthocenter to each vertex. The center of the 9-point circle, labeled *N*, is collinear with the triangle's centroid, circumcenter, and orthocenter (which all sit on the triangle's **Euler line**). It is located one-quarter of the way from the centroid to the orthocenter.



Can you prove the facts we've encountered so far?

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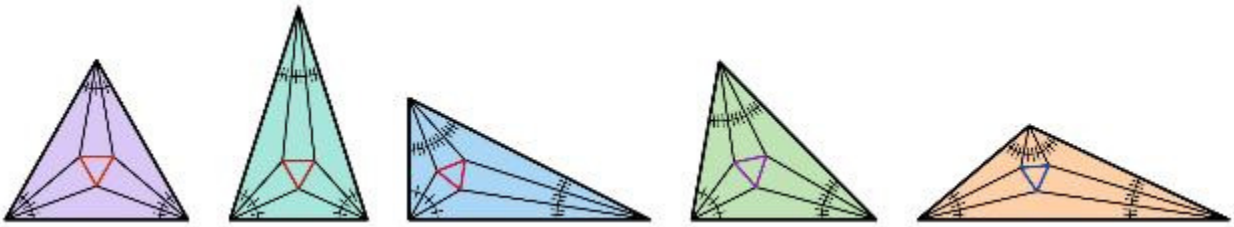
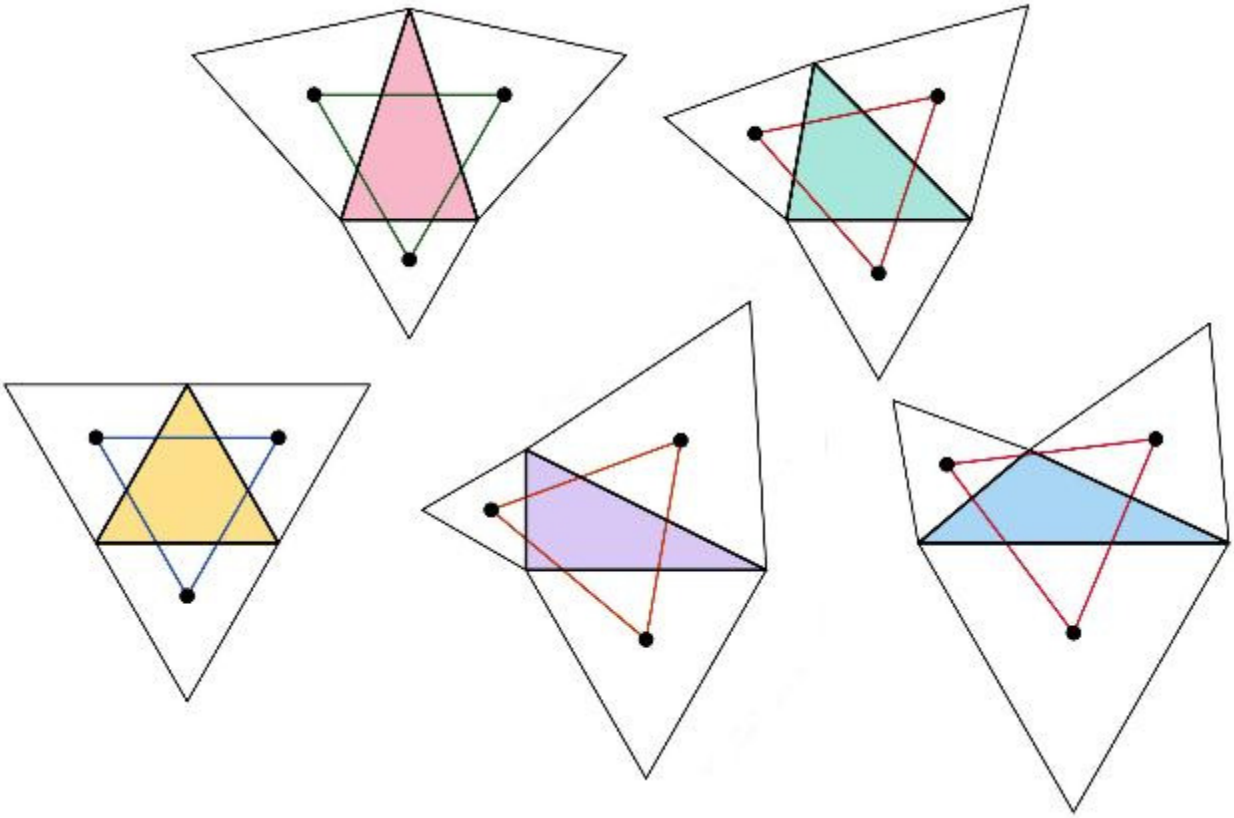
Surprisingly, in 1899, Frank Morley discovered that the angle trisectors next to each side meet in pairs at the vertices of an equilateral triangle, now called the **Morley triangle**:

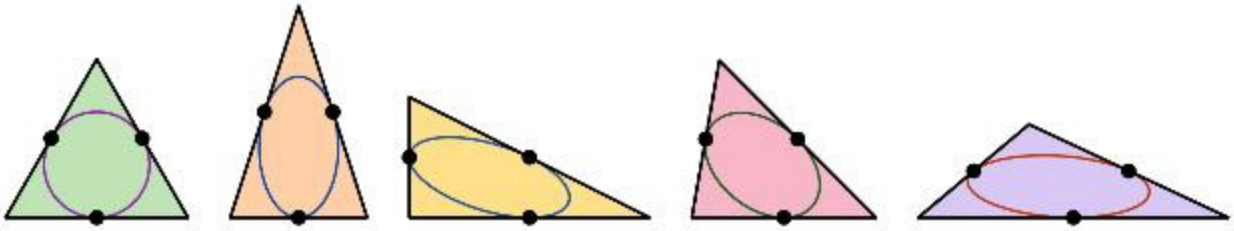


There are many triangles that are associated with a given triangle, such as the **Napoleon triangle**:



Can you see how the equilateral triangle (indicated by the dots) is constructed?

Finally, let's take a look at the **Steiner inellipse**:



This is the unique ellipse that is tangent to the sides of the triangle exactly at the midpoints of each side.

Can you invent your own triangle associated concept? If you do, let us know!

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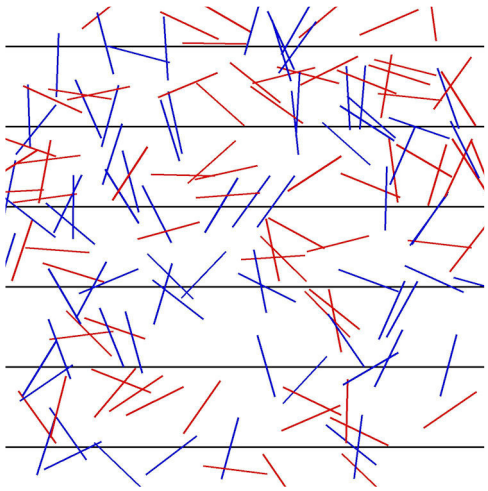
Learn by Doing

Spaghetti: Raw and Cooked

by Girls' Angle Staff

Note: Problems vary in level of involvement. You'll find that you can do some quickly, but others (in red) are intended to give you something to think about for days. You're always welcome to email us or ask a mentor about this at the club.

In this Learn by Doing, we'll present Buffon's needle problem and Barbier's generalization of it, Buffon's noodle.



150 needles of length 2/3″ with parallel line spacing 5/6″. Blue needles cross a parallel. This gives the approximation $\pi \approx 3.2$.

Let's start with Buffon's needle problem. We take a plane and cover it with equally spaced, mutually parallel lines. Let the separation distance between consecutive parallel lines be D. Onto this plane, we toss a needle (or raw piece of spaghetti) of length L. For now, we'll assume that $L < D$. It will land on the plane at some random location and pointing in some random direction.

Let x be the distance of the center of the needle to the nearest parallel line, and let A be the acute angle measure (in radians) that the needle makes with the parallel lines. We assume that x is uniformly distributed between 0 and $D/2$ and A is uniformly distributed between 0 and $\pi/2$, and that x and A are independent.

1. What is the probability that the needle intersects a parallel line? If you're having trouble with this problem, move on to Problem 2; otherwise, skip to Problem 5.

2. Fix A. Show that the needle intersects a parallel line exactly when $x \le (L/2)\sin A$.

Consider the rectangular patch of the A-x coordinate plane where $0 \le A \le \pi/2$ and $0 \le x \le D/2$. Because x and A are independent and uniformly distributed, the probability that (A, x) lies in a region R of the rectangle is the area of R divided by the area of the rectangle.

3. In this rectangular patch, shade the points that correspond to the needle crossing one of the parallel lines.

4. Find the fraction of the area of the rectangular patch that the region you shaded in Problem 3 occupies.

Spoiler Alert!

For Problem 4, did you get $\dfrac{2L}{\pi D}$?

5. What is the probability that the needle intersects a parallel line if $L > D$?

6. What is the expected number of intersections of the needle with the parallel lines (for any L)?

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Now suppose our needle is no longer in the shape of a line segment but is shaped like some curve in the plane of total length L. (That is, someone cooks the spaghetti, lets it fall loosely on the plate, and then lets it dry and harden. Assume such a noodle falls so that the distance between its center of mass and the nearest parallel and the orientation (from 0 to 2π radians) it makes are uniformly and independently distributed.) We'll assume that the curved shape of our noodle can be arbitrarily well approximated by a piecewise linear path.

7. What is the expected number of intersections of our curved noodle with the parallel lines? If you're having trouble with this problem, move on to Problem 8; otherwise, skip to Problem 12.

Recall that if X and Y are random variables, then the expected value of $X + Y$ is the sum of the expected values of X and Y separately, irrespective of whether X and Y are dependent or not.

8. Suppose we take two copies of our noodle and stick them end-to-end. If E is the expected number of intersections between one randomly placed noodle and the parallel lines, what is the expected number of intersections between this noodle of twice the length and the parallel lines?

9. Explain how Problem 8 can be generalized to show that E is proportional to L. (Hint: Do this for line segments of any length first, then approximate the noodle with a piecewise-linear path.)

Thus, $E = cL$ for some constant c.

Instead of using integration to compute c, we use a clever idea following Barbier (1860):

10. Consider a circle of diameter D. How many intersections will there be between the circle and the parallel lines?

11. Using Problem 10, what must the constant c be?

12. Suppose that R is a convex region of constant width $W \le D$. (That is, if you roll the shape on a flat surface, it can carry a platform without causing the platform to bob up and down as it rolls. For example, the Reuleaux triangle, which is formed by taking an equilateral triangle then replacing each of its edges with a circular arc centered at the opposite vertex and with the same endpoints, has this property. It is the best-known region of constant width other than the circle. There are infinitely many others.) Exploit the constant width to determine the probability that R intersects a parallel line. How many times do you expect its perimeter to intersect with a parallel line? Conclude that the perimeter of R is πW.

For a challenge, try to turn the argument around:

13. (Crofton's formula) In the xy-coordinate plane, let C be a curve with arclength L. Let $n(\varphi, p)$, for $0 \le \varphi \le \pi$ and p any real number, be the number of intersections of C with the line whose equation is $x \cos \varphi + y \sin \varphi = p$ (i.e., the line whose signed distance from the origin is p and which points in the direction $\varphi - \pi/2$). Show that

$$L = \frac{1}{2}\iint n(\varphi, p)\, d\varphi dp\,.$$

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Notes from the Club

These notes cover some of what happened at Girls' Angle meets. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

Session 39 - Meet 1
September 10, 2026

Mentors: Elisabeth Bullock, Elsa Frankel, Layla Jarrahy, Victoria Li, Yaqi Li, Laura Pierson, Molly Roughan, Amanda Tran

We welcome all new and returning members and mentors to the start of our 20th year of Girls' Angle!

Some members told us about a counting game they had played at school. Players take turns counting upward from 1, and on your turn, you must say either one or two numbers. For example, the first player might say "1, 2," the second, "3," the first again, "4, 5," and so on. Whoever says the target number wins. With two players and a target of 10, would you rather go first or second? What is your strategy? What about any target N? What if you may say up to k numbers on each turn? What if whoever says the target number loses? What if you may say either one or three numbers, but not two? And what happens with three players? Can any one of them guarantee a win?

Session 39 - Meet 2
September 17, 2026

Mentors: Elisabeth Bullock, Elsa Frankel, Layla Jarrahy, Yaqi Li, Tatiana Medved, Hanna Mularczyk, Laura Pierson, Molly Roughan

One member couldn't remember how a trick for multiplying by 11 went, so she checked her guesses using the standard multiplication algorithm, and in doing so, she saw what the trick had to be. Tricks are easy to forget, but if you understand why something works, you can reinvent it whenever you need it.

Here are the squares of the first several numbers that end in 5:

$$5^2 = 25,\ 15^2 = 225,\ 25^2 = 625,\ 35^2 = 1225,\ 45^2 = 2025,\ 55^2 = 3025,\ 65^2 = 4225.$$

One member noticed that the differences between consecutive squares in this list, 200, 400, 600, 800, …, grow by 200 each time. Does the same thing happen for squares of numbers that end in 1? In 7? Members split up the digits 0 through 9 among themselves and checked them all. Can you explain why it always works? Can you use the list above to find a fast way to square any number that ends in 5? What happens if you look at cubes instead of squares? Notice that $45^2 = 2025$, which was last year. What is the next year that will be a perfect square?

Session 39 - Meet 3
September 24, 2026

Mentors: Elisabeth Bullock, Ashley Forkey, Victoria Li, Yaqi Li, Hanna Mularczyk, Laura Pierson, Molly Roughan, Anlan Xu

A member who is studying linear algebra was given a matrix puzzle that we'll state here without matrices. Start with the Pythagorean triple (3, 4, 5). Given any triple (a, b, c), form the new triple

$$(a + 2b + 2c,\ 2a + b + 2c,\ 2a + 2b + 3c).$$

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Applied to (3, 4, 5), this gives (21, 20, 29), and indeed, $21^2 + 20^2 = 29^2$. Before applying the rule, you may first change the sign of a or of b. For instance, applying the rule to (–3, 4, 5) gives (15, 8, 17), and applying it to (3, –4, 5) gives (5, 12, 13). So each triple gives birth to three new ones.

Can you prove that the rule always produces a Pythagorean triple? Call a Pythagorean triple primitive if its three numbers share no common factor bigger than 1. Are the new triples always primitive? Draw the family tree of triples for a few generations. Which leg length is listed first? Up to the order of the legs, does every primitive triple appear somewhere in the tree? Does any appear twice? What happens if both a and b are negative before applying the rule?

Session 39 - Meet 4
October 1, 2026

Mentors: Elisabeth Bullock, Elsa Frankel, Layla Jarrahy, Victoria Li, Yaqi Li, Tatiana Medved, Hanna Mularczyk, Laura Pierson, Molly Roughan, Anlan Xu

Visitor: Avery Vilandrie, Shiftsmart

Avery Vilandrie, a Girls' Angle member back in 2010–11, returned as part of our Support Network to tell us about her work at Shiftsmart, where she builds models that predict sales. Avery graduated from Brown University in 2023 with a degree in applied math and economics.

She began by showing how quickly a computer can add up the whole numbers from 1 to 100, and then from 1 to 1,000,000. (Can you beat the computer by hand?) Then she described her work: she gathers data from over 4,000 gas station convenience stores and predicts how many hot dogs each will sell on a given day, so that the stores know how many to order and make. Using a toy example of lemonade sales versus foot traffic in a store, she plotted data, found a line of best fit using linear regression, and used the line to make predictions. In the Q&A, she described using sales at nearby stores to suggest what a store should stock. She also told us about her years as a competitive skier. When asked about AI, she said that it is useful, but a person still has to check the final results, and that when you're learning, it's important to struggle with problems yourself.

Here's a regression problem to try: Of all lines $y = mx + b$, which one minimizes the sum of the squares of the vertical distances to the points (1, 2), (2, 3), and (3, 7)? Why use squares of distances, rather than the distances themselves?

At the same meet, a question arose: How can you fold a strip of paper into exact thirds? Here's one way. Make a guess crease roughly a third of the way from the left end. Fold the right end over to the guess crease and make a new crease. Then fold the left end over to the new crease and make a third crease. Take this third crease as your new guess, and repeat. The creases rapidly home in on the one-third mark. Why? If your first guess is off by an inch, how far off is your guess after one round? Can you devise similar methods for folding fifths? Sevenths? Which fractions can you fold this way?

Session 39 - Meet 5
October 8, 2026

Mentors: Ashley Forkey, Elsa Frankel, Layla Jarrahy, Victoria Li, Tatiana Medved, Hanna Mularczyk, Laura Pierson, Molly Roughan, Anlan Xu

Members folding origami asked for "fun facts," and one that delighted them was about paper itself. Most of the world uses paper sizes named A0, A1, A2, and so on. An A0 sheet has an area of exactly 1 square meter, and when you cut an An sheet in half parallel to its short side, you get two A(n + 1) sheets that have exactly the same shape as the original. What must the ratio of the sides be? What are the dimensions of an A4 sheet, to the nearest millimeter? Photocopiers often offer settings that enlarge by 141% and reduce by 71%. Why those percentages?

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Session 39 - Meet 6 October 15, 2026	Mentors:	Ashley Forkey, Elsa Frankel, Victoria Li, Hanna Mularczyk, Laura Pierson, Molly Roughan, Anlan Xu

Members experimented with adding sine waves on Desmos, which brings us to this identity:

$$\sin(at) + \sin(bt) = 2\sin((a+b)t/2)\cos((a-b)t/2).$$

Graph $y = \sin(20x) + \sin(22x)$. What do you see? How does the identity explain the picture? The slowly varying factor, $\cos(x)$, has period 2π, yet the swells in the graph come twice as often. Why? Musicians tune their instruments by listening for such swells in loudness, called "beats." If one string vibrates at 440 Hz and another at 442 Hz, how many beats per second will you hear?

At another table, a mentor folded an origami pig out of yellow paper. When members objected that pigs are pink, she offered this "proof" that all pigs are yellow. We show that in any group of n pigs, all the pigs are the same color. If $n = 1$, there's nothing to prove. Suppose it is true for every group of n pigs, and take a group of $n + 1$ pigs. Set one pig aside; the other n are the same color. Return it and set aside a different pig; again, the other n are the same color. Because the two groups overlap, all $n + 1$ pigs are the same color. Since there is a yellow pig (look at the origami one!), all pigs are yellow. This is a variant of the classic 'all horses are the same color' argument, which goes back to George Pólya, who posed it in 1954 as 'any n girls have eyes of the same color.' By the end of the meet, members found the flaw.

Session 39 - Meet 7 October 22, 2026	Mentors:	Elsa Frankel, Layla Jarrahy, Victoria Li, Tatiana Medved, Hanna Mularczyk, Laura Pierson, Molly Roughan

Some members tried unscrambling proofs: the sentences of a proof are shuffled, and your job is to put them back in order. Interestingly, members put the word "Proof." at the very end, as if to announce that a proof had just been completed. (Mathematicians write it at the start and often mark the end with a small square, ■.) Here is a scrambled proof that there are infinitely many primes. Can you unscramble it?

(A) Then N is greater than 1, so N has at least one prime factor. Call it q.
(B) This contradicts our assumption, so there must be infinitely many primes.
(C) Let $N = p_1p_2 \cdots p_k + 1$.
(D) But q also divides N, so q divides $N - p_1p_2 \cdots p_k = 1$, which is impossible.
(E) Suppose there are only finitely many primes, and list them all: $p_1, p_2, \ldots, p_k$.
(F) Since q is prime, it is on our list, so q divides the product $p_1p_2 \cdots p_k$.

Is there more than one correct order? Which sentence holds the key idea? What makes a scrambled proof easy or hard to put back together?

Session 39 - Meet 8 October 29, 2026	Mentors:	Elsa Frankel, Victoria Li, Tatiana Medved, Hanna Mularczyk, Laura Pierson, Molly Roughan

For some mysterious reason, the powers of 7 have arisen at the club this month in three completely different contexts. Do powers of 7 mean anything special to you?

Can you think of an algorithm that produces arbitrarily long sequences of zeros and ones in such a way that the sequence appears to be generated by picking zero or one uniformly and independently at random?

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Calendar

Session 39: (all dates in 2026)

September	10	Start of the thirty-ninth session!
	17	
	24	
October	1	Avery Vilandrie, Shiftsmart
	8	
	15	
	22	
	29	
November	5	
	12	
	19	
	26	Thanksgiving - No meet
December	3	

Session 40: (all dates in 2027)

January	28	Start of the fortieth session!
February	4	
	11	
	18	No meet
	25	
March	4	
	11	
	18	
	25	No meet
April	1	
	8	
	15	
	22	No meet
	29	
May	6	

Girls' Angle has run nearly 200 Math Collaborations. Math Collaborations are fun, fully collaborative, math events that can be adapted to a variety of group sizes and skill levels. We now have versions where all can participate remotely. We have now run four such "all-virtual" Math Collaborations. If interested, contact us at girlsangle@gmail.com. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

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Girls' Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) _____ (first) _____

Parents/Guardians: _____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: _____ Cell Phone: _____

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

The $50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- ☐ Enclosed is a check for $50 for a 1-year Girls' Angle Membership.
- ☐ I am making a tax-deductible donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

A Math Club for Girls

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Girls' Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is $20/meet for members and $30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle is based in Cambridge, Massachusetts. For security reasons, only members and their parents/guardians will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

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Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, founder and director of the Exploratory
Yaim Cooper, Institute for Advanced Study
Julia Elisenda Grigsby, professor of mathematics, Boston College
Kay Kirkpatrick, professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, assistant dean and director teaching & learning, Stanford University
Lauren McGough, senior research scientist, University of Chicago
Mia Minnes, teaching professor, CSE, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls
Elissa Ozanne, associate professor, University of Utah School of Medicine
Katherine Paur, The Nueva School
Bjorn Poonen, professor of mathematics, MIT
Liz Simon, graduate student, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, professor of mathematics, University of Washington
Karen Willcox, Director, Oden Institute for Computational Engineering and Sciences, UT Austin
Lauren Williams, professor of mathematics, Harvard University

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high-level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

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Girls' Angle: Club Enrollment Form

Applicant's Name: (last) _____ (first) _____

Parents/Guardians: _____

Address: _____ Zip Code: _____

Home Phone: _____ Cell Phone: _____ Email: _____

Please fill out the information in this box.

Emergency contact name and number: _____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

_____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

_____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

_____ Date: _____
(Parent/Guardian Signature)

Participant Signature: _____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is $216 for one session (12 meets)	☐ I will pay on a per meet basis at $30/meet.
☐ I will pay on a per meet basis at $20/meet.	☐ I'm including $50 to become a member, and I have selected an item from the left.

☐ I am making a tax-deductible donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

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Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls
Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

_____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: _____Date: _____

Print name of applicant/parent: _____

Print name(s) of child(ren) in program: _____

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