Girls' Angle Bulletin

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To Foster and Nurture Girls' Interest in Mathematics

An Interview with Mirjana Vuletić
The Search for Heron Triangles
The Twin Fives That Pollard's Rho Can't Pull Apart
Taylor Series for Tangent, Part 5

Summer Fun Solutions:
Hexaflexagons!
World Cup Summer Fun!
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From the Founder

We're entering Girls' Angle's 20th year and are looking forward to the 120th issue of the Bulletin, our 481st meet, and, especially, to working with our members, new and returning. Who knows what math will flow out of their minds? I'm eager to see! -Ken Fan, President and Founder

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Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: Heron and Schubert Triangles, by C. Kenneth Fan. For more, see page **Error! Bookmark not defined.**. Claude AI assisted with the generation of this image.

An Interview with Mirjana Vuletić

Being a mathematician is like having a sixth sense. At first glance, math seems like a human creation… But even in its most abstract forms, math seems inevitable and feels more like something that is discovered rather than invented.

Mirjana Vuletić is Associate Professor of Mathematics at UMass Boston. She earned her Bachelor of Science and Master of Science degrees from the University of Novi Sad in Serbia, followed by a doctoral degree in mathematics from Caltech under the supervision of Alexei Borodin. She is an expert on plane partitions, which she explores as both combinatorial objects and random models.

Ken: When did you become interested in mathematics? What was it that got you interested?

Mirjana: My sister, who is five years older than me, was my inspiration. (She might not know that she is the one to blame.) She was in eighth grade when she competed in a national math competition. I don't know what got her interested, but seeing her do it, I told myself that I would try to compete too.

I started participating in competitions the following year when I was in fourth grade. As I progressed through middle school and high school, math became even more fascinating. This was especially thanks to the many college professors who worked with us outside of school and introduced us to interesting problems and research.

Ken: Do you have a favorite math problem or mathematical concept from your youth that you can share with us?

Mirjana: This isn't necessarily my favorite problem, but it's the one I remember most vividly from my early days in math: Given a square grid, in how many ways can a fly travel from the bottom-left corner to the top-right corner if it can only move one step up or one step to the right?

Of course, this is a classic binomial coefficient problem, but at the time I had no idea how to solve it. I remember talking about it with my teacher as he was driving me home from the competition. He teased me for not being able to solve such an "easy" problem and threatened to kick me out of the car (he didn't!).

Ken: You earned your Bachelor of Science and Master of Science degrees from the University of Novi Sad in Serbia. Are there important ways in which math education in Serbia differs from that in the United States?

Mirjana: One of the biggest differences is the strong math community that exists in Serbia. Many mathematicians generously devote their time to working with talented students. They encourage young people to engage in math together as a shared social experience. That sense of belonging was a very important part of my own mathematical development and a great motivator.

My own children, who were raised in the US, have experienced this sense of belonging through music and sports; they have been encouraged by the community to pursue these endeavors and to develop their skills and enthusiasm for them together with their peers. I wish young people had the same opportunity to discover and pursue math.

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Thank you and best wishes,
Ken Fan
President and Founder
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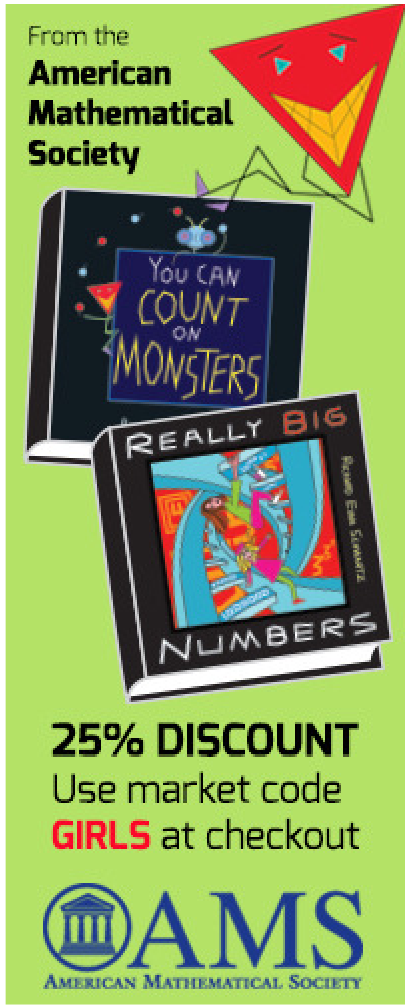

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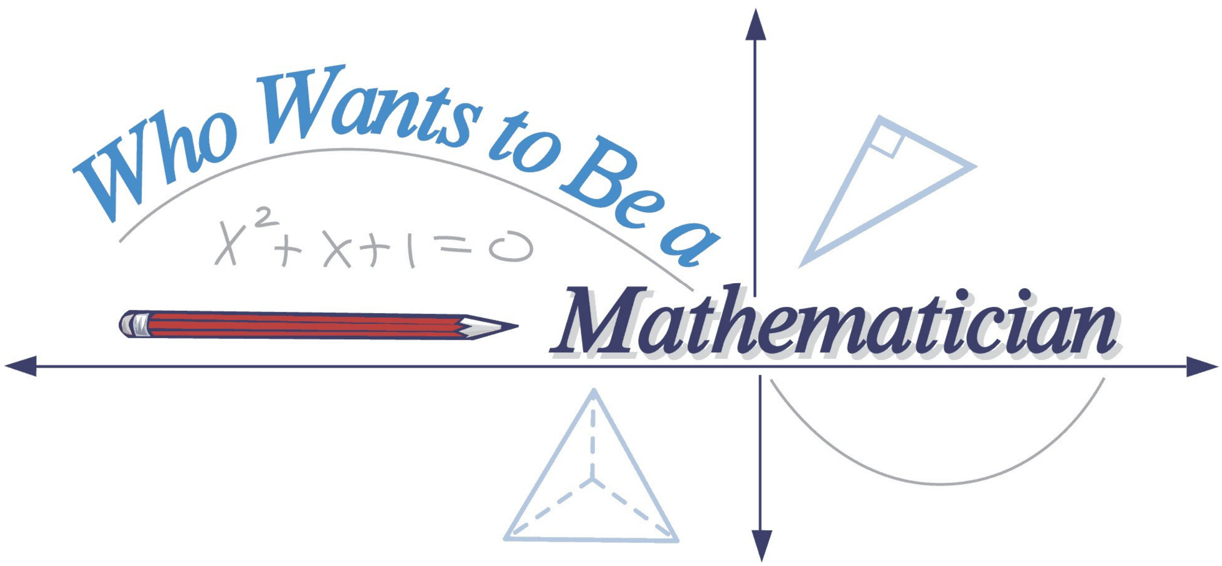

The Search for Heron Triangles[1]

by Lindsey Watson | edited by Amanda Galtman

Lindsey is a former Girls' Angle member. Last year, she graduated from the University of East Anglia with a degree in mathematics.

How does one calculate the area of a triangle? The first thought that might come to mind is the formula $\frac{1}{2}bh$, with base side length b and height h. You might not be aware there exists an equation that calculates area purely from the side lengths (a, b, c). This is immensely helpful when the height of a triangle is not immediately known. This equation is called Heron's formula:

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}\,, \qquad (1)$$

where Δ is the area and $s = \frac{a+b+c}{2}$ is the semiperimeter. When integer sides produce an integer area, the triangle is known as a Heron triangle.

Exercise 1. Let's consider the triangles with side lengths (5, 5, 6) and (4, 5, 7). Applying Heron's formula, calculate their areas. Both have integer side lengths, but are they both Heron triangles? What is different in each case?

Asking which integers correspond to side lengths of a Heron triangle is equivalent to asking which choices of a, b, c cause $s(s-a)(s-b)(s-c)$ to be a perfect square.

Now you might wonder, what happens if we scale a Heron triangle?

Exercise 2. A (3, 4, 5) triangle has area 6. What is the area of a (6, 8, 10) triangle? What about a (9, 12, 15) triangle? What pattern do you notice between the scale factor of the sides and the scale factor of the area?

This scaling relationship gives us some useful freedom. Any triangle with rational side lengths and rational area is similar to a Heron triangle.

A natural next step would be to ask: how can we choose different combinations of integer side lengths to guarantee integer area? Using only Heron's formula, how can we go about searching for these Heron triangles? Can we simply test triples (a, b, c) one at a time? What if we want to find them all?

We could try combinations of numbers, but there is a more efficient approach, which is where parameterization comes in. Parameterization simplifies complex shapes and solution sets by expressing a system in terms of independent variables called parameters. In the world of Heronian mathematics, parameterization allows for the generation of all Heron triangles. There is more than one method, and different parameterizations can reveal different features of a triangle. Let's explore two different approaches.

[1] This work is based on Lindsey's Year-Long Math Project supervised by Shaun Stevens.

Method 1: **Brahmagupta & Euler Parameterization.** The Indian mathematician Brahmagupta described one of the earliest known parameterizations for generating Heron triangles in the 7th century. Later, this was developed more formally by the Swiss mathematician Euler in the 18th century. Although their original derivations are not fully known, it is suspected they used Bachet's idea of juxtaposing two right triangles along a common altitude, the line drawn from a vertex that meets the opposite side at a right angle. This construction gives a natural way to generate infinitely many Heron triangles.

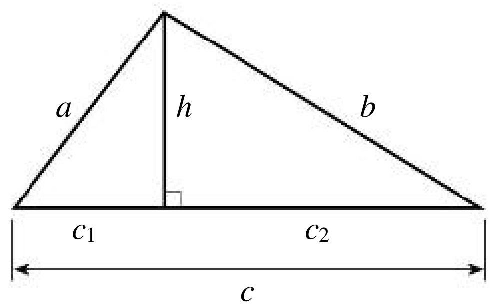

Figure 1. Juxtaposition of Right Triangles

Note that while the juxtaposed triangles can be placed back to back, they can also be overlapped, often in a position where the altitude is not interior to the triangle. Before we dive into the derivation, let's consider the triangle (13, 14, 15). Dropping the altitude of length 12, we get two right triangles, (9, 12, 15) and (5, 12, 13). Placing these triangles back to back gives a base of 14. What happens if instead we overlap the two right triangles? Subsequently, determining the base becomes a matter of subtraction rather than addition.

We can now generalize this construction. Let h be the common altitude, and let c_1 and c_2 be the two segments of the base. In our (13, 14, 15) example, $h = 12$, $c_1 = 9$, and $c_2 = 5$. From Figure 1, we see $c_1 + c_2 = c$. Applying the Pythagorean theorem to the two right triangles formed by altitude h, we get $a^2 = h^2 + c_1^2$ and $b^2 = h^2 + c_2^2$. Rearranging, we get

$$h^2 = a^2 - c_1^2 = b^2 - c_2^2. \qquad (2)$$

We want to find positive rational side lengths a, b, and c that satisfy (2), simplifying the search for rational triangles. Factoring (2), we get

$$h^2 = (a + c_1)(a - c_1) = (b + c_2)(b - c_2).$$

Let's isolate the first part, starting with $h^2 = (a + c_1)(a - c_1)$. We need two rational numbers whose product is h^2. Let's name one of them m, and then the other must be h^2/m. Repeating for $h^2 = (b + c_2)(b - c_2)$, we name one rational factor n. Therefore,

$$a + c_1 = m,\ a - c_1 = h^2/m.$$

Adding these equations eliminates c_1, giving

$$2a = h^2/m + m,$$

so

$$a = \frac{1}{2}\left(\frac{h^2}{m} + m\right).$$

Following this process for b and then adding the base segments $c_1 + c_2$ gives

$$a = \frac{1}{2}\left(\frac{h^2}{m} + m\right),\ b = \frac{1}{2}\left(\frac{h^2}{n} + n\right), \text{ and } c = \frac{1}{2}\left(m + n - \frac{h^2}{m} - \frac{h^2}{n}\right).$$

These expressions are rational, but we already saw that scaling a triangle does not change its shape, so we can clear the denominators by multiplying all sides by $2mn$.

After we reassign a, b, and c to the scaled lengths, our parameterization becomes

$$a = n(m^2 + h^2),\ b = m(n^2 + h^2),\ c = (m + n)(mn - h^2). \tag{3}$$

Now notice the $mn - h^2$ factor in c. What happens if $mn = h^2$? Then we get $c = 0$, so the triangle collapses to a degenerate case. Thus, we require $mn > h^2$. Assuming the triangle is not degenerate, what is its area? Recalling that the scaled altitude is $2hmn$, can you find the area without Heron's formula?

Now we've shown that the sides of any rational triangle are proportional to a, b, and c in (3), for some m, n, and h, by a rational factor, with the resulting area,

$$\Delta = hmn(m + n)(mn - h^2). \tag{4}$$

Starting from two right triangles, we've just converted the search for Heron triangles into a choice of only three rational parameters h, m, n.

Exercise 3. Let's experiment a bit. Let $h = 1$, $m = 2$, $n = 3$. What triangle does (3) produce? Use (4) to find the area. Do you recognize what family it belongs to?

Exercise 4. What happens if h, m, n are all doubled? Can you predict how the side lengths and area will scale before doing the calculation?

Further Challenges

Exercise 5. Heron's formula contains neither a height nor an angle. How is this so? Try deriving (1) to understand. First, think about it trigonometrically using the sine area formula and the law of cosines. Then, try dropping an altitude and using the Pythagorean theorem. What similarities do you notice between the approaches?

Exercise 6. Using (1), prove that the area of every Heron triangle is divisible by 6. Hint: Start by thinking about divisibility by 2 and 3.

Exercise 7. Applying (4), prove by contradiction that there is no Heron triangle with a side length of 1 or 2.

Method 2: Schubert's Parameterization. Heron triangles have integer, and therefore rational, side lengths and area, but must their medians, which are the line segments connecting each vertex to the midpoint of the opposite side, be rational as well? Schubert investigated this question on medians and developed a parameterization for Heron triangles with a rational median. He conjectured that no Heron triangle could have two rational medians, but triangle (73, 51, 26) later provided a counterexample. Interestingly, no Heron triangle with three rational medians is currently known, and whether one exists remains an open problem.

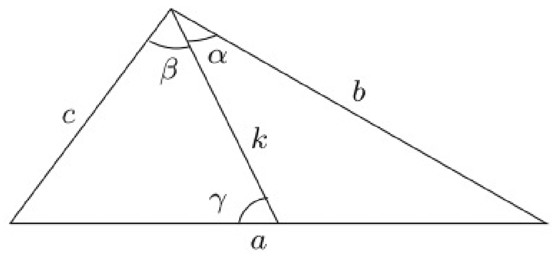

Figure 2. Triangle with Labeled Median k

Before we launch into Schubert's approach, pay attention to why a median divides a triangle into two triangles of equal area: the base a is split into two equal side lengths. Now let's introduce median k, and mark the angles $\angle\alpha$, $\angle\beta$, and $\angle\gamma$. We want to relate the side lengths and median to these angles, which can be done using the sine area formula $\Delta = \frac{1}{2}ab \sin C$. In this case, for the three angles formed by median k, we get

$$\Delta = bk \sin \alpha = ck \sin \beta = \tfrac{1}{2} ak \sin \gamma. \tag{5}$$

Angles $\angle\alpha$, $\angle\beta$, and $\angle\gamma$ are not independent because they arise from the same median. We need a single relationship to connect them, and a convenient way to find one is to view the sides and median as vectors. From Figure 2, a can be written as a difference of two side vectors $\boldsymbol{a} = \boldsymbol{b} - \boldsymbol{c}$. Taking the dot product with $\boldsymbol{k}$, we get $ak \cos \gamma = bk \cos \alpha - ck \cos \beta$. Dividing by the area using (5), we get

$$2 \cot \gamma = \cot \alpha - \cot \beta. \tag{6}$$

Exercise 8. Suppose that $\alpha = \beta$. What does the area relation in (5) tell you about b and c? What does (6) tell you about γ? Does this align with what you know about medians in an isosceles triangle?

The goal is to translate (6) into a rational, algebraic expression. How can we replace the angles with rational parameters? The equation (6) still depends on angles. Schubert's clever observation was that translating those angles using the half-angle cotangents allows (6) to be written using rational operations. We define the Schubert parameters as

$$M = \cot(\alpha/2),\ P = \cot(\beta/2),\ X = \cot(\gamma/2). \tag{7}$$

Since $M = \cot(\alpha/2)$, we have $1/M = \tan(\alpha/2)$, so $\cot \alpha = \frac{1}{2}(M - 1/M)$. Following the same process for P and X, we can then substitute into (6) to get

$$M - \frac{1}{M} = P - \frac{1}{P} + 2(X - \frac{1}{X}).$$

Clearing the denominators, we produce what's known as the Schubert surface:

$$2MP(X^2 - 1) + MX(P^2 - 1) - PX(M^2 - 1) = 0. \tag{8}$$

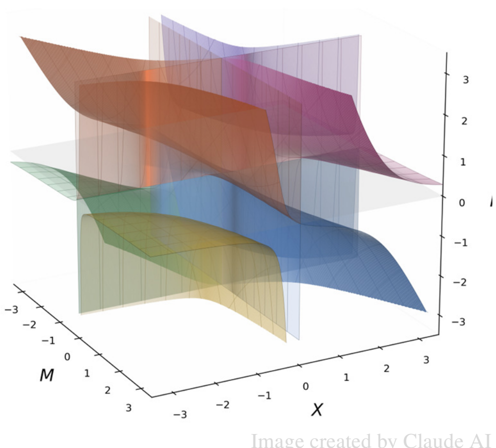

Figure 3. Graph of the Schubert Surface

We've managed to transform a problem about triangles into a problem about points in three-dimensional space. Instead of searching for Heron triangles with rational medians, we can look for rational triples $(M, P, X) \in \mathbb{Q}^3$ satisfying (8).

Exercise 9. Suppose $X = 1$. What does this tell you about γ? Substitute $X = 1$ into the Schubert surface, and factor. What relationship do you find between M and P? How does this connect to Exercise 8?

We can also express the Schubert parameters (7) in terms of the side lengths of the triangle. Using the law of cosines and Apollonius' theorem (see the equation in Exercise 11), we can write the parameters (M, P, X) in terms of the side lengths, area, and median, as seen below in (9). So, every Heron triangle with a rational median gives a rational point on the Schubert surface.

$$M = \frac{4\Delta}{4bk + a^2 - 3b^2 - c^2}, \quad P = \frac{4\Delta}{4ck + a^2 - b^2 - 3c^2}, \quad X = \frac{4\Delta}{2ak - b^2 + c^2}. \tag{9}$$

As you might expect, the angles of a Heron triangle can be found using parameters M, P, and X, but the actual side lengths cannot be identified until a scale is chosen. We know the full shape once we know

$$a : b : c = \frac{a}{c} : \frac{b}{c} : 1.$$

Using (5),

$$\frac{a}{2c} = \frac{\sin\beta}{\sin\gamma}, \quad \frac{b}{c} = \frac{\sin\beta}{\sin\alpha}. \tag{10}$$

So $\frac{a}{c} = 2\frac{\sin\beta}{\sin\gamma}$. Using the same half-angle identities as before, we can write (10) in terms of the Schubert parameters:

$$\frac{a}{c} = \frac{2(X + X^{-1})}{P + P^{-1}}, \quad \frac{b}{c} = \frac{M + M^{-1}}{P + P^{-1}}. \tag{11}$$

So M, P, and X determine $a : c$ and $b : c$. Therefore, the Schubert parameters determine the ratio $a : b : c$ and hence the shape of the triangle.
Exercise 10. What happens to the Schubert parameters if all sides of a triangle are doubled? Why does this mean that a point on the Schubert surface determines the shape rather than the actual size of a Heron triangle?

Schubert's parameterization does not provide a direct formula for generating side lengths and area in the same way Brahmagupta/Euler's procedure does. Instead, Schubert's parameterization establishes a correspondence between rational points on the Schubert surface and similarity classes of Heron triangles with a rational median, where the chosen median k determines the corresponding point.

One Last Challenge

Exercise 11. Can you recover one of Schubert's parameters (9) from the triangle? Start with $M = \cot(\alpha/2)$, for example. Use the sine area formula and the law of cosines to find $\sin\alpha$ and $\cos\alpha$. Then, apply Apollonius' theorem,

$$b^2 + c^2 = 2\left(k^2 + \left(\frac{a}{2}\right)^2\right),$$

to eliminate k^2.

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The Twin Fives That Pollard's Rho Can't Pull Apart

by Isabella Li | edited by Amanda Galtman

Isabella Li is a rising senior at Lexington High School.

This summer, I stood next to my poster at the International Congress of Mathematicians and told strangers about the time my code refused to factor five numbers. That refusal turned out to be the best thing on the poster, and it started with a mistake I almost did not catch.

I spent last year comparing classical factoring algorithms against one another, timing how long each took to fully factor the same numbers. Factoring here means the complete job: given a number N, break it all the way down into primes, like $2047 = 23 \times 89$. That distinction will matter later. It is easy for small numbers and famously hard for big ones, which is roughly why encryption can keep your credit card number safe when you buy something online[2]. My test set included the Mersenne numbers, the numbers of the form $2^i - 1$. Some of them are prime, like $2^5 - 1 = 31$, and those need no factoring at all. The interesting ones are composite, like $2^{11} - 1 = 2047$.

Two of my five algorithms were invented by the same person, John Pollard, one year apart: Pollard's $p - 1$ in 1974 and Pollard's rho (ρ) in 1975. For most of the project I believed they were one algorithm wearing two names! In my defense, the Greek letter rho, ρ, looks like a p.

Then, while preparing my poster, I sat down to check that every claim on it matched something my code actually did. It is the kind of chore you do with a snack, expecting to find nothing. Halfway through, I stopped. Every result I had labeled "$p - 1$" had actually been computed by ρ. I had been racing four algorithms, not five, and one of them was wearing the other's name tag.

My first thought was that if this one label was wrong, maybe my entire project was wrong. That would only turn into a bigger problem the longer I left it alone. This mistake was so agitating that I stayed up two hours over my bedtime to correct everything. I cracked open a fresh bottle of grape electrolyte drink and worked through the old and new files, one anxious pass at a time. Later that night, I still wasn't able to figure out how one method had ended up running under two different names in my paper, so I kept going back to it for the rest of the week.

What the fix required was concrete enough: implement the real $p - 1$ algorithm, correct every label, and rerun the whole comparison. Rerunning meant going back through my own output, line by line instead of skimming, and that is when the mystery turned up. I had been staring at those tables for months without really looking at them.

Before the mystery, though, I owe you the rho algorithm itself. Starting with a number N, set $x_0 = 1$ and $x_{i+1} = x_i^2 + 1 \pmod{N}$, for $i \geq 0$.

For each step, compute $\mathrm{GCD}(|x_{2i} - x_i|, N)$. Usually, it is 1, which tells you nothing and means you have to keep building the sequence. Occasionally, it is N itself, which means this run is a dead end. In that case, you start over with a different x_0 or, as 25 is about to show, with a different polynomial. Any other GCD value is a divisor of N, and that is a win.

[2] The connection is RSA encryption, whose security rests on the difficulty of factoring large numbers.

The x_i sequence is the orbit of the function $x \to x^2 + 1$ on $\mathbf{Z}/N\mathbf{Z}$. There are only N values to land on, so sooner or later the sequence hits one it has visited before. From then on, it is stuck repeating that stretch forever. Every orbit takes on the same shape: a tail you traverse once, and then a loop you never get out of. That structure is like a line curling into a circle, which is a lowercase ρ, and that is where the name, rho, comes from.

So why does taking a greatest common divisor find anything? Take p to be one of the prime factors of N. The algorithm will never know p, but we can still reason about it from the outside. Reducing everything modulo p does not disturb the sequence, and what changes is how much room the sequence has. Modulo N, it can visit at most N different values before it is forced to repeat, but modulo p, there are only p of them, and p is far smaller. The smaller sequence therefore closes its loop long before the bigger one does. Usually, in the gap between those two moments, there is an index i with

$$x_{2i} = x_i \pmod{p} \text{ but } x_{2i} \neq x_i \pmod{N}.$$

Their difference is then a multiple of p and not a multiple of N, so the greatest common divisor of that difference with N is a factor. That word *usually* is important. Sometimes, the two loops close at exactly the same moment, and afterward the greatest common divisor is always 1 or N, forever. The rest of this article is about a number where that is exactly what happens.

The strange part is that the smaller loop closes somewhere you are not able to look. You do not know p, so you cannot check anything modulo p; p is the entire thing you are hunting for. You just watch the greatest common divisor and let it notice for you.

I wanted full prime factorizations, and the method always hands back one divisor at a time. So, I ran it again on that divisor and on whatever was left, and again on the pieces of those, until nothing composite remained.

The rho method completely factored $2^i - 1$ for every i from 2 to 107, with exactly five exceptions: $i = 20, 40, 60, 80$, and 100. For those five, it pulled out prime factor after prime factor and then froze, with one small piece left that it could not break in two. What made that strange is that these are not difficult numbers. The number $2^{20} - 1$ is $3 \times 5^2 \times 11 \times 31 \times 41$. Its largest prime factor is only 41, and small factors are supposed to be rho's specialty.

How long rho takes depends mostly on the smallest prime factor p: modulo p, the sequence has only p values to wander through, so it can be expected to revisit one within roughly $\sqrt{p}$ steps. Naturally, I thought this method would shine when a number has small factors, which these do. The one method built for this situation was the only one that could not get through it. I wrote the five exponents in a column and stared at them. Every twentieth one. A pattern that clean is rarely an accident, so I went hunting for the reason.

Step one was to look at the leftover pieces. They were the same number all five times: 25.

Why would 25 show up in exactly every twentieth Mersenne number? It's because of the order of 2 modulo 25, meaning the smallest $k > 0$ with $2^k = 1 \pmod{25}$. That order divides $\varphi(25) = 20$, where φ is the Euler phi-function, so it is one of 1, 2, 4, 5, 10, or 20. Every one of those below 20 divides either 10 or 4, so it is enough to check those two: $2^{10} = 1024 = 24 \pmod{25}$ and $2^4 = 16$, neither of which is 1 modulo 25. So the order is exactly 20, which says that 25 divides $2^i - 1$ precisely when 20 divides i. My five stubborn numbers were exactly the ones carrying a factor of 25. And none of this is really about Mersenne numbers; hand the algorithm any number with a 25 inside it and the same thing goes wrong.

Step two: what does ρ have against 25? Isolating a factor of 25 was no trouble at all; my runs did it every time. The trouble came at the end, when 25 was all that was left and the two 5s inside it had to come apart. (The number $2^{100} - 1$ actually carries 5^3, so there the run split off a single 5 first and then stalled on the remaining 25.)

When we are running the algorithm with $N = 25$, splitting it requires an index i with

$$x_{2i} = x_i \pmod 5 \text{ and } x_{2i} \neq x_i \pmod{25}.$$

That way, the difference is divisible by 5 but not by 25, and the greatest common divisor is exactly 5. If the two values are congruent modulo 25, the greatest common divisor is 25 and tells us nothing new. If they are not congruent modulo 5, the greatest common divisor is 1.

So I traced the orbit of $x \to x^2 + 1$ on $\mathbf{Z}/25\mathbf{Z}$ by hand. The algorithm's typical start, $x_0 = 1$, is already sitting on the cycle, so begin at 0 instead if you want to see the tail. Starting at 0: you get 1, then 2, then 5, then 26 = 1 (mod 25), which puts you back where you were three steps ago. That little cycle, $1 \to 2 \to 5 \to 1$, is the whole story. Start at 3 instead: 10, then 101 = 1 (mod 25), and you are in the same cycle. I tried all 25 starting values, and every one of them lands in that cycle after a few steps.

Now look at the three values in the cycle: 1, 2, and 5. Modulo 5 they are 1, 2, and 0, so no two of them are congruent modulo 5. That means once both x_i and x_{2i} are on the cycle, which happens almost immediately, being congruent modulo 5 forces them to be equal modulo 25. The case we need simply does not occur, and checking all 25 starting values, including the few steps of tail, confirms it: the greatest common divisor is always 1 or 25, never 5. The two 5s inside 25 are twins that the polynomial $x^2 + 1$ cannot pull apart, and restarting does not rescue you, because every starting value funnels into the same cycle.

Here's a picture to go with the congruences that I used while working it out. Think of the 25 values as houses sorted into five neighborhoods, where two houses share a neighborhood exactly when they are congruent modulo 5. One neighborhood is {3, 8, 13, 18, 23}. The method wins when its two values land in the same neighborhood but in different houses. However, that never happens with the cycle 1, 2, 5, which puts one value in each of three different neighborhoods.

Naturally, I wanted to know whether 25 was unusual. For every odd prime p up to 97, I checked whether any starting value lets $x^2 + 1$ split p^2. For every one of them except $p = 5$, at least one start works. For $p = 5$, none of the 25 starts does. (The starting value matters here: 13, 17, and 41 stall with $x_0 = 1$, but a different start rescues them.)

And the real $p - 1$, the method missing from my first race all along? It cracked all five stubborn numbers instantly. Its trick: build k, a multiple of $p - 1$, whenever $p - 1$ is built entirely from small prime powers. (For example, take $k = \text{LCM}(1, \ldots, 7^2)$.) For almost any a, Fermat's little theorem gives $a^k = 1 \pmod p$, so $\text{GCD}(a^k - 1, N)$ reveals p. For the prime 5, $p - 1 = 4 = 2^2$, about as small as a number gets, so k caught it on the very first pass. Then it turned around and failed on three other Mersenne numbers, for a completely different reason. Two methods named after the same creator, and their blind spots do not overlap at all. That contrast became the centerpiece of my poster, and none of it would exist if I had not caught my own mistake.

A few weeks later, I asked the obvious follow-up question. If the rule $x \to x^2 + 1$ is the problem, does changing the rule help? Swap in $x \to x^2 + 2$, and 25 splits immediately. But when I reran my five stubborn numbers with $x \to x^2 + 2$, all five got stuck again, and four of them stalled at $15 = 3 \times 5$. Here is the reason, and it is worth doing concretely. Modulo 15, the rule $x \to x^2 + 2$ has exactly two cycles, $3 \to 11 \to 3$ and $6 \to 8 \to 6$, both of length two. Subtract the two values inside either cycle and you get 8 or 2, and neither of those shares a factor with 15. So once x_i and x_{2i} are both on a cycle, their difference is either 0 or coprime to 15, and the greatest common divisor is either 15 or 1. Splitting 15 would require the two values to agree modulo 3 while differing modulo 5, or the reverse, and that never happens. The reason is the lengths. Modulo 3 and modulo 5, every cycle has length two and every tail is at most two steps, so in both pictures, x_i and x_{2i} agree exactly at the even steps from step two onward. Those are the same steps, so the values agree modulo 3 precisely when they agree modulo 5. The tails themselves need not match; starting from 0, the sequence is already on its cycle modulo 3 but takes two steps to get there modulo 5. All that matters is that both tails are short enough for step two to be past them.

The fifth number, $2^{60} - 1$, stalled at 45 instead. It is the only one of the five divisible by 9, and $x \to x^2 + 2$ cannot separate the two 3s inside 9, exactly the way $x \to x^2 + 1$ cannot separate the two 5s inside 25.

Changing the rule did not remove the blind spot. It moved it. That, I later learned, is exactly why the standard advice when ρ fails is to restart with a fresh polynomial rather than the same one. The advice sounds like superstition, but it is not: switching polynomials is how you avoid that perfect agreement.

Things to try:

1. Pick any starting value from 0 to 24 and iterate $x \to x^2 + 1$ modulo 25. In how many steps do you reach the cycle $1 \to 2 \to 5$? What is the largest number of steps any starting value needs?

2. Use $x \to x^2 + 2$ modulo 25 instead. Find two values in the orbit that are congruent modulo 5 but not modulo 25. Their difference has greatest common divisor exactly 5 with 25, which is how this rule succeeds where the other one fails.

3. Iterate $x \to x^2 + 2$ modulo 15 from a few different starting values, writing x_i and x_{2i} side by side. At each step, check whether they are congruent modulo 3 and whether they are congruent modulo 5. The answer should always be the same for both, which is why no greatest common divisor ever lands strictly between 1 and 15. Start at 0 to see that the two sequences need not reach their cycles at the same step.

I used to think a mistake in a project was something to erase. Now I think a mistake with a pattern in it is just a theorem you have not met yet.

Taylor Series for Tangent, Part 5

by Ken Fan | edited by Jennifer Sidney

The nth derivative of $\tan(x)$, with respect to x, is

$$\frac{p_n(2\sin(x))}{\cos^{n+1}(x)},$$

where $p_n(x)$ is defined recursively by $p_1(x) = 1$ and

$$p_{n+1}(x) = \frac{(n+1)xp_n(x) + (4 - x^2)p'_n(x)}{2},$$

for $n \geq 1$, where $p'_n(x)$ denotes the derivative of $p_n(x)$ with respect to x. Let $c_{k,n}$ be the coefficient of x^k in $p_n(x)$. We also define $c_{-1,n}$ to be 0. We have

$$c_{k,n+1} = \frac{n-k+2}{2}c_{k-1,n} + 2(k+1)c_{k+1,n}.$$

Jasmine: Pemmican is good!

Emily: I'm not sure I like it. I think it could use some blueberries.

Jasmine: I can see that, but I like it just the way it is. It's like a meat brownie.

Emily: It seems like we're discovering all kinds of things about the tangent function *except* for a way to easily compute its Taylor series! That whole business with the $d_{k,n}$ was amusing, but it feels like a big detour.

Jasmine: It does. Let's go back to our original recursion relation for the polynomials $p_n(x)$:

$$p_{n+1}(x) = \frac{(n+1)xp_n(x) + (4 - x^2)p'_n(x)}{2},$$

for $n \geq 1$, where $p_1(x) = 1$ and $p'_n(x)$ denotes the derivative of $p_n(x)$ with respect to x. We really want to know $p_n(0)$, but having that derivative in the recursion formula complicates matters. Maybe we should abandon this $p_n(x)$ approach?

Emily: It's too bad x isn't just equal to 2.

Jasmine: Why?

Emily: If $x = 2$, then the derivative term disappears.

Jasmine: That's amusing! But, of course, x isn't equal to 2. Although the recursion sure does simplify a lot at $x = 2$. It becomes

$$p_{n+1}(2) = (n+1)p_n(2).$$

Emily: Wait a sec! Isn't that just the recursion relation for factorial?

Jasmine: It is! Well, it is if $p_1(2) = 1$. Is it?

Emily: $p_1(x) = 1$, so $p_1(2)$ is 1… just what we need for $p_n(2)$ to be n factorial!

Jasmine: Gosh, yet another amusing fact:

$$n! = \sum_{k=0}^{n-1} c_{k,n} 2^k .$$

That's a funny connection between factorial and the tangent function. What do you make of it?

Emily: I'm not sure. We defined the $p_n(x)$ by the formula

$$\tan^{(n)}(x) = \frac{p_n(2\sin(x))}{\cos^{n+1}(x)},$$

and $2\sin(x) = 2$ when $x = \pi/2$. But $x = \pi/2$ is where $\tan(x)$ has a vertical asymptote and $\cos(x)$ is zero, so we wouldn't have been able to see the factorial identity from this nth derivative of tangent formula.

Jasmine: At tangent's pole on the other side, $x = -\pi/2$, the derivative term in the recurrence relation also vanishes. The recurrence relation becomes $p_{n+1}(-2) = -(n+1)p_n(-2)$, so similarly to $p_n(2)$, we have $p_n(-2) = (-1)^{n+1} n!$.

Emily: We can also see that from the fact that $p_n(x)$ has terms of degrees that are all the same parity. So when n is odd, $p_n(-2) = +n!$, and when n is even, $p_n(-2) = -n!$.

Jasmine: That's a quick way to see it!

Emily: Gee. We set out on a quest to find the Taylor expansion of the tangent function, but ended up finding two families of coefficients and proving some nifty facts about them; yet I feel no closer to finding that Taylor expansion!

Jasmine: Hey, Emily – I just had a weird thought. The coefficients $c_{k,n}$ are all nonnegative integers. And now we know that if we sum them up weighted by the powers of 2, we get $n!$. The main context in which I know $n!$ is as the number of permutations of 1 through n. Do you think that the $c_{k,n}$ might have some combinatorial interpretation in terms of permutations?

Emily: Whoa … that would be absolutely amazing! Let me just make sure I'm hearing you right. We now know that $n! = 2^{n-1}c_{n-1,n} + 2^{n-3}c_{n-3,n} + 2^{n-5}c_{n-5,n} + \ldots + 2^{n-1-2k}c_{n-2-2k,n} + \ldots$, which is the number of permutations of 1 through n. So you're wondering if the permutations naturally partition into sets of size $2^{n-1-2k}c_{n-1-2k,n}$ for each k?

Jasmine: Yes, that's exactly what I'm wondering. Could it be?

Emily: Let's see if we can realize such a partition for small n. If it's true, I think we should be able to do that, with the partition for permutations on 1 through $n+1$ being built from that of the permutations on 1 through n in accordance with the recursion relation.

Jasmine: Good plan! I've got the first few polynomials $p_n(x)$ here in my notes. They are: 1, x, $x^2 + 2$, $x^3 + 8x$, and $x^4 + 22x^2 + 16$. Each term of each polynomial should be matched with some subset of permutations. Specifically, a term of the form ax^k should correspond to a subset with $a2^k$ permutations.

Emily: Let's let S_n be the set of permutations on the numbers 1 through n. In S_1, there's only the one permutation that sends 1 to itself, and $p_1(x) = 1$, so at least that matches nicely. But S_2 has two permutations, which are 12 and 21 in one-line notation. So if what you're saying is true, all of S_2 would be the partition, and that would have to be a clue as to what permutations should be put together in the same part.

Jasmine: With $p_3(x) = x^2 + 2$, we finally get a partition with more than one part. We have to partition S_3 into two parts, one of size 2 and the other of size $2^2 = 4$. A priori, there are 6 choose 2, or 15, ways to form such a partition. How can we possibly identify which is the "right" one?

Emily: I'm not sure, but 15 isn't too big. Let's list them all and see what they look like!

Jasmine: Okay!

Emily and Jasmine create this table:

	Part (size 2)	**Part (size 4)**
1	123 132	213 231 312 321
2	123 213	132 231 312 321
3	123 231	132 213 312 321
4	123 312	132 213 231 321
5	123 321	132 213 231 312
6	132 213	123 231 312 321
7	132 231	123 213 312 321
8	132 312	123 213 231 321
9	132 321	123 213 231 312
10	213 231	123 132 312 321
11	213 312	123 132 231 321
12	213 321	123 132 231 312
13	231 312	123 132 213 321
14	231 321	123 132 213 312
15	312 321	123 132 213 231

Emily: Fifteen is pretty big! It feels like an ocean of possible partitions. Which one of these are we supposed to pick?

Jasmine: Gosh, I really have no idea…

To be continued …

Summer Fun!

In the previous issue, we presented the 2026 Summer Fun problem sets.

In this issue, we give solutions to many of the problems. Our solutions may be terse and, in some cases, are more of a hint than a solution. We prefer not to give detailed solutions before we know that most of the members have spent time thinking about the problems. The reason is that *doing* mathematics is very important if you want to learn mathematics well. If you haven't tried to solve these problems yourself, you won't gain as much when you read these solutions.

If you haven't thought about the problems, we urge you to do so *before* reading the solutions. Even if you cannot solve a problem, you will benefit from trying. By working on the problem, you will force yourself to think about the associated ideas. You will gain familiarity with the related concepts and that will make it easier and more meaningful to read others' solutions.

With mathematics, don't be passive! Be active!

Move your pencil and move your mind – you might discover something new.

Also, the solutions presented are *not* definitive. Try to improve them or find different solutions.

Solutions that are especially terse will be indicated in red. Please do not get frustrated if you read a solution and have difficulty understanding it. If you run into difficulties, we are here to help! Just ask!

Please refer to the previous issue for the problems.

Members: *Don't forget that you are more than welcome to email us with your questions and solutions!*

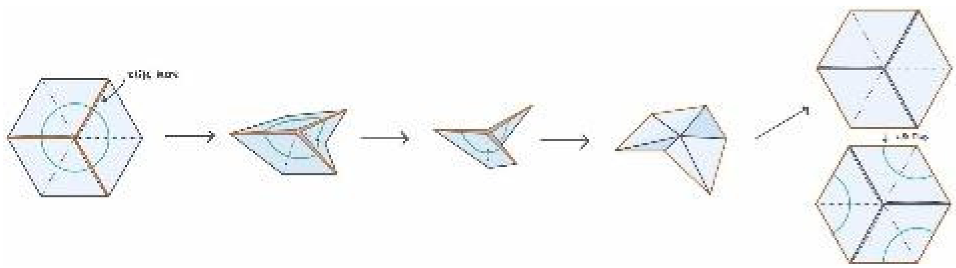
Summer Fun!

Hexaflexagons!

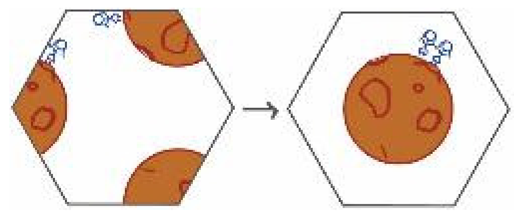
By AnaMaria Perez

1. If you pinch and open the hexaflexagon twice without flipping it, you'll find all three faces! To make sure they're all different, try labeling each one with a 1, 2 or 3 when you find them.

2. If you draw a circle in the center, then flex the hexaflexagon once and flip it, the circle will be broken into 3 different arcs.



You can also work the other way around: draw the broken circle, then flex to bring them together – perhaps, to tell the story of two friends stranded on different planets finally able to reunite!



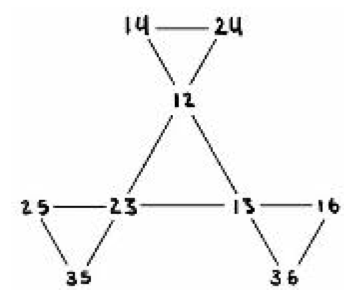
3. Before folding, draw a lengthwise line down the strip's center, end to end, on both sides. Fold and tape your hexaflexagon as usual, then cut along the line you drew. You'll get a single band even though you've cut it down the middle! That is because the hexaflexagon is "twisted" each time you fold it over itself and therefore has 3 twists. Play around with how many pieces you get when you twist a paper different numbers of times before taping the ends!

4. This hexaflexagon has 6 faces. This should make sense: the strip you used had 19 triangles, and the two triangles at the end of the strip get glued together to make a single triangle. There are then 18 triangles, each with a front and back, giving us 36 triangular faces. Each face of the hexaflexagon uses 6 triangles, so in total, we should have 6 faces. See how efficiently hexaflexagons make use of space? The only surfaces you cannot write on are the two faces you glue together.

5. Start by lettering your faces A, B, C, D, E and F. I'll use the numbers 1-6 in my solution; once you see which of your faces behaves like my face 1, you can match up the rest. You've noticed by now that to fold a hexaflexagon to find a new face, we must pinch 3 of the 6 radii of the hexagon. In the 3-faced hexaflexagon, there is only one way to do this. However, you might notice that **sometimes** for the 6-faced hexaflexagon there are two ways to do this!

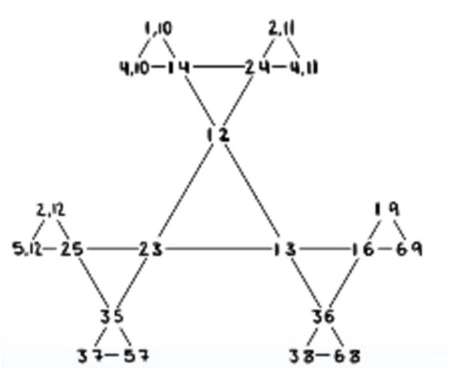
Summer Fun!

The two ways are highlighted in orange and blue. Now, if you pinch either set of radii and it can open up to a new face, it will do so with ease. If this doesn't happen, don't try to force it! You might break your hexaflexagon! We want a way to go from any face to any other face very quickly. Perhaps when you were trying to label your hexaflexagon you found the 6th face to be very elusive. At any given moment, your hexaflexagon can be described by the two faces showing, the *front* face and the *back* face. We will write the state of the hexaflexagon as an ordered pair (Front, Back), or, when we don't need to specify which is front and which is back, as a two-digit number with the lower face number first (e.g., 12 for the pair {1, 2}). For example, (2, 1) describes the hexaflexagon with front face 2 and back face 1, and 12 could mean either (2, 1) or (1, 2). Of course there are more faces hiding inside, but we'll focus on describing our hexaflexagon by what we can see. Based on the faces showing, we know exactly which faces we can get to by pinching and opening exactly once. For example, if 1 is in front and 2 is in back, i.e. we're in state (1, 2), you can pinch and fold it in two ways to get to faces 3 or 4. If you start with 1 on top you will go from (1, 2) to either (1, 3) or (1, 4). If you start with 2 on top, you will go from (2, 1) to (2, 3) or (2, 4). With my labeling, the following pairs are possible (up to switching front and back): 12, 13, 14, 16, 23, 24, 25, 35, 36. You will notice that, with the way I numbered my hexaflexagon, faces 1, 2 and 3 are easy to find, whereas faces 4, 5 and 6 don't come up quite as often. Furthermore, two "high numbers" can never both be showing. From here, we can make a graph describing which pairs are connected by a single pinch and fold operation:



Now you see that to find the elusive 6th face, you need only get to state 13, then flex along both sets of radii until you reach 6!

6. You can build larger hexaflexagons by gluing together more pieces of paper. By wrapping the paper as we did to create the 6-faced hexaflexagon, you halve the number of visible triangles, making it look like the template for a smaller hexaflexagon. So we need to attach 2^n pieces of paper with 10 triangles each (overlapping and gluing one triangle at each join) to make a $3(2^n)$-faced hexaflexagon. We can continue wrapping again and again until we reach something that looks like the original template to create hexaflexagons with more and more faces! We can repeat Problem 5 for the 12-faced hexaflexagon and get a similar graph. Notice the nice fractal structure! You might be able to guess what the graph will look like as we keep making our hexaflexagons with more faces ;).



Summer Fun!

World Cup Summer Fun!

By Ella Wilson

2. Because of the symmetries of a sphere, any point can act as the "North Pole" if you rotate the sphere itself. Let's continue now with this globe analogy. If we put our string at the North Pole, pick any other location on the globe, and stretch our string tight to meet it, we have just drawn the line of longitude on which that location lies. All lines of longitude go through both the north and south poles and split the globe exactly in half. Therefore, this shortest path that we have just drawn is actually an arc along a great circle!

3. Let's once again use our globe analogy. Say that our lune has one vertex at the North Pole. Then the second vertex of our lune will have to be at the South Pole. Because we can rotate our sphere however we want, no matter where the first vertex of a lune is, the second vertex will always be the point on the sphere directly opposite, or *antipodal*, to the first vertex. Therefore, both side lengths will then be πR where R is the radius of our sphere. Further, this symmetry gives us that the two angles where the lines intersect at the opposite poles are identical. Since all side lengths are equal and the interior angles are the same, every lune is a regular polygon.

4. This is false for spherical geometry! As we showed above, all lunes are regular polygons and we can have lunes with different interior angles. For example, think about the lines of longitude. Every pair of longitude lines will form a lune, but they can have varying angles of intersection.

5. Let the vertex angle of the lune be θ (in radians). The total angle completely around a pole will still be 2π. The area of the lune will take up a fraction of the sphere equal to $\theta/2\pi$ of the surface area of the sphere. Multiplying this fraction by the total surface area of the sphere we get

$$\left(\frac{\theta}{2\pi}\right)\times 4\pi R^2 = 2\theta R^2.$$

6. One example that we can create is a triangle where every angle is actually a right angle, measuring $\pi/2$ radians. To see this, draw a lune with angle measurement $\pi/2$. Then take the midpoints of both sides of the lune and draw a segment connecting them. This segment will perpendicularly bisect both of these sides (split them in half and intersect at a right angle). We now have two equilateral triangles where every angle is $\pi/2$.

Further, using the properties we explained in the problem statement and the solution to Problem 8, we will have that every spherical triangle will have to have angle sum strictly less than 3π (otherwise it would encompass an entire hemisphere) and we will get that a spherical triangle always has angle sum greater than π.

Summer Fun!

7. Extending the three sides into full great circles divides the sphere into exactly 8 distinct spherical triangles. Each spherical triangle will be contained within exactly 3 lunes (each corresponding to one of its interior angles).

8. *Proof.* Notice first that the triangle is contained completely within three lunes, one with angle measure α, one with β, and one with γ. Additionally, after we extended the sides of the triangle, we got a congruent triangle $\Delta A'B'C'$ directly opposite from ΔABC and $\Delta A'B'C'$ will also be completely contained within three lunes of angle measures α, β, and γ respectively as these lunes are the antipodal counterparts to the lunes containing ΔABC. From Problem 5, we have that the lunes have area $2\alpha R^2$, $2\beta R^2$, and $2\gamma R^2$. The sum of the areas of all six of these lunes is the surface area of the sphere, with both triangles counted three times. Therefore, we have that

$$2(2\alpha R^2 + 2\beta R^2 + 2\gamma R^2) = 4\pi R^2 + 2\text{Area}(\Delta ABC) + 2\text{Area}(\Delta A'B'C').$$

Since ΔABC is congruent to $\Delta A'B'C'$, this simplifies to Girard's Theorem:

$$\text{Area}(\Delta ABC) = R^2(\alpha + \beta + \gamma - \pi). \qquad \square$$

9. From Girard's Theorem, we don't actually need to do heavy trigonometry for every combination! Because angle sum is determined by how much the angle sum exceeds π, we can rule out a number of triples of cities that clearly can't have the maximum or minimum area, although we still have to be careful to really single out the maximum and minimum. We found:

- The three cities that encompass the largest geographic area/have the largest sum of interior angles are Vancouver, Boston, and Mexico City.
- By looking up more accurate distances, it turns out that the three cities that encompass the smallest geographic area/have the smallest sum of interior angles form the flattest triangle and are Atlanta, New York/NJ, and Mexico City. (From the given data, 11 triples appear to line up on the same great circle, which would give zero area.)

If you did use trigonometry to get exact angle measurements, you should have seen that even the largest sum of interior angles still gives us a sum of angles close to π or 180°, which really speaks to how big the Earth is!

10. The standard soccer ball tiling is not the only way in which you can tile a sphere using regular pentagons and hexagons! Here are some observations we can make about the properties of a tiling of this type:

Recall what we found in Problem 6: spherical triangles and thus polygons will always bulge! So, their interior angles must be strictly greater than their flat Euclidean counterparts. Regular spherical pentagons have to have interior angle measures strictly greater than $3\pi/5$ radians and regular spherical hexagons have to have interior angle measures strictly greater than $2\pi/3$. We need to have more than two polygons meeting at a single vertex (otherwise we would have a lune), and so we can never have three hexagons meeting at a vertex.

Summer Fun!

There is a formula called Euler's formula which, applied to tilings of the sphere, tells us that the number of vertices, V, minus the number of edges, E, plus the number of faces, F, must equal 2, i.e., $V - E + F = 2$. Let p be the number of pentagons and h the number of hexagons. Note that around a vertex, there must be 3 spherical polygons because there must be more than 2 and the smallest interior angle for a regular pentagon or hexagon must exceed $3\pi/5$, and $4(3\pi/5) = 12\pi/5 > 2\pi$. Therefore, our number of vertices would be $(5p + 6h)/3$, our number of edges would be $(5p + 6h)/2$, and our number of faces would be $p + h$. Applying Euler's formula, we know that

$$\frac{5p+6h}{3} - \frac{5p+6h}{2} + p + h = 2$$

and when we simplify this expression we get $p = 12$. Every tiling of the sphere that uses only pentagons and hexagons will always have exactly 12 pentagons on it!

You can actually tile the sphere using only 12 pentagons (see the next problem for a fuller explanation of why this works!).

11. You should get exactly six types of monohedral tilings. Five will be fixed tilings that correspond with the Platonic solids:

- 4 spherical triangles with angle measure $2\pi/3$ — corresponds to a tetrahedron
- 6 spherical squares with angle measure $2\pi/3$ — corresponds to a cube
- 8 spherical triangles with angle measure $\pi/2$ — corresponds to an octahedron
- 12 spherical pentagons with angle measure $2\pi/3$ — corresponds to a dodecahedron
- 20 spherical triangles with angle measure $2\pi/5$ — corresponds to an icosahedron

The final type is the family of monohedral tilings that you can make from lunes. At each vertex, the total angles meeting at that point must be exactly equal to 2π. So, we can tile the sphere with a positive integer number of lunes by using lunes with angle measurement $2\pi/n$ for $n \geq 2$.

Because the spherical polygons bulge and for non-lunes we must have at least three meeting at each vertex, we can rule out any regular polygon with six or more sides from making a monohedral tiling. We can play this angle measurement game for all of our other tilings as well. For triangles, we must have that the angle measurement is greater than $\pi/3$ and that at each vertex we have an integer number of triangles. We know that $2\pi/3$, $2\pi/4 = \pi/2$, and $2\pi/5$ are then our only options for angle measurements. Similarly for the squares and pentagons we must have that the angle measurement is strictly greater than $\pi/2$ and $3\pi/5$ respectively, so our only option for both of these would be to have angle measurements of $2\pi/3$. After drawing these out on your sphere, you should see that all of these give you monohedral tilings!

Frieze, Please!

By Hanna Mularczyk

For Problem 10, if you hunted down a frieze pattern that you particularly like, please take a photo and send it to us at girlsangle@gmail.com!

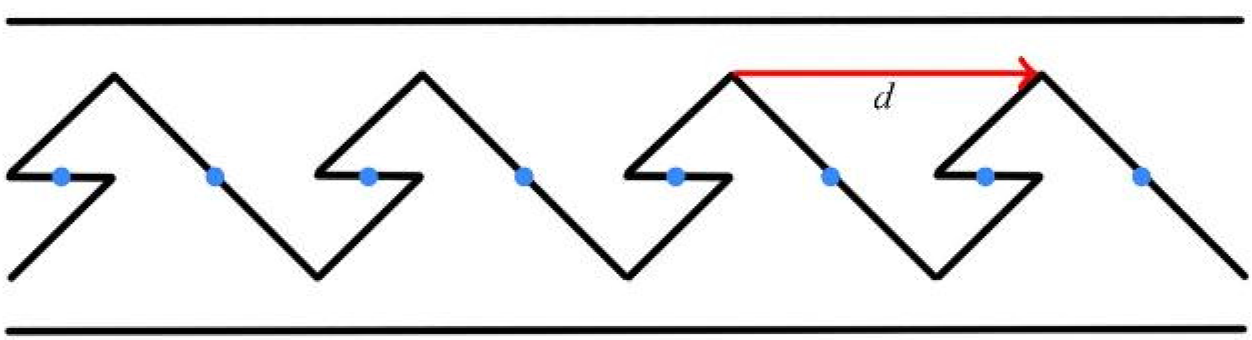
1. The composition of two symmetries is the composition of two transformations, which is also a transformation. Each symmetry doesn't change the pattern, so we still end up with the same pattern. Thus, the composition is a transformation that doesn't change the pattern, which is a symmetry.

2. The only reflections that send a blank strip to itself are a reflection across the horizontal line halfway through the strip and a reflection across any vertical line on the strip.

3. The only rotations that send a blank strip to itself are rotations by 180° with the center of rotation on the horizontal line halfway through the strip.

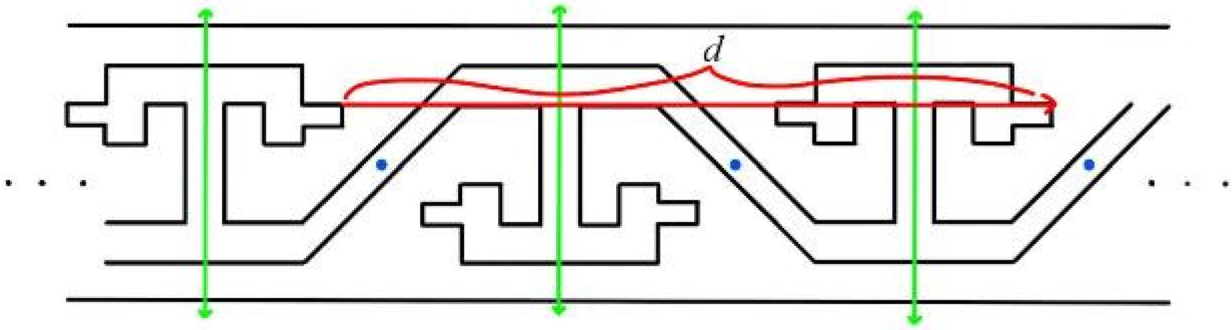
4. A reflection across the horizontal line would flip the dog upside down, a reflection across a vertical line would make the dog face left instead of right, and a 180° rotation would also put the dog upside down (in a different way), so none of these can map this pattern to itself.

5. The smallest distance of translation is marked as *d.* The symmetry group contains no reflections, and it contains rotations by 180° at any of the blue dots.

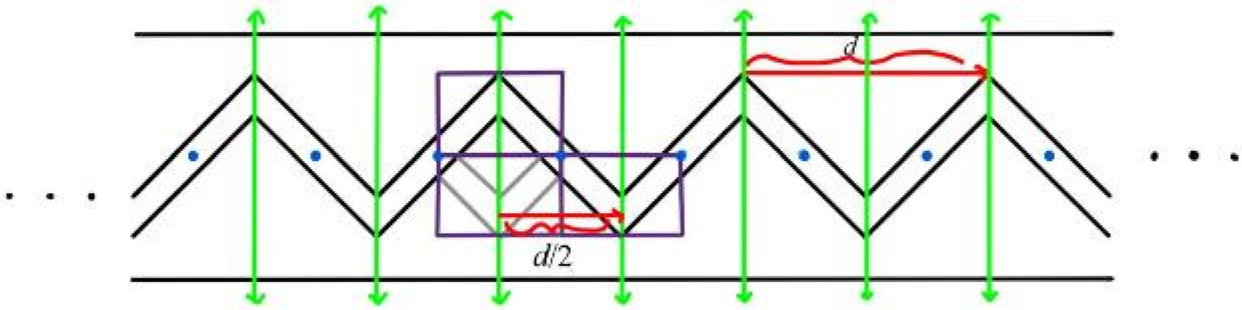


Summer Fun!

6. The smallest distance of translation is marked as *d*. The symmetry group contains reflections across any of the green vertical lines and rotations by 180° at any of the blue dots.



7. This frieze pattern's symmetry group contains translation by the distance marked *d*, reflections across any of the green vertical lines, rotations by 180° at any of the blue dots, and a glide reflection by distance *d*/2, shown by the purple boxes. These are the same symmetries as the previous frieze, so they have the same symmetry group.

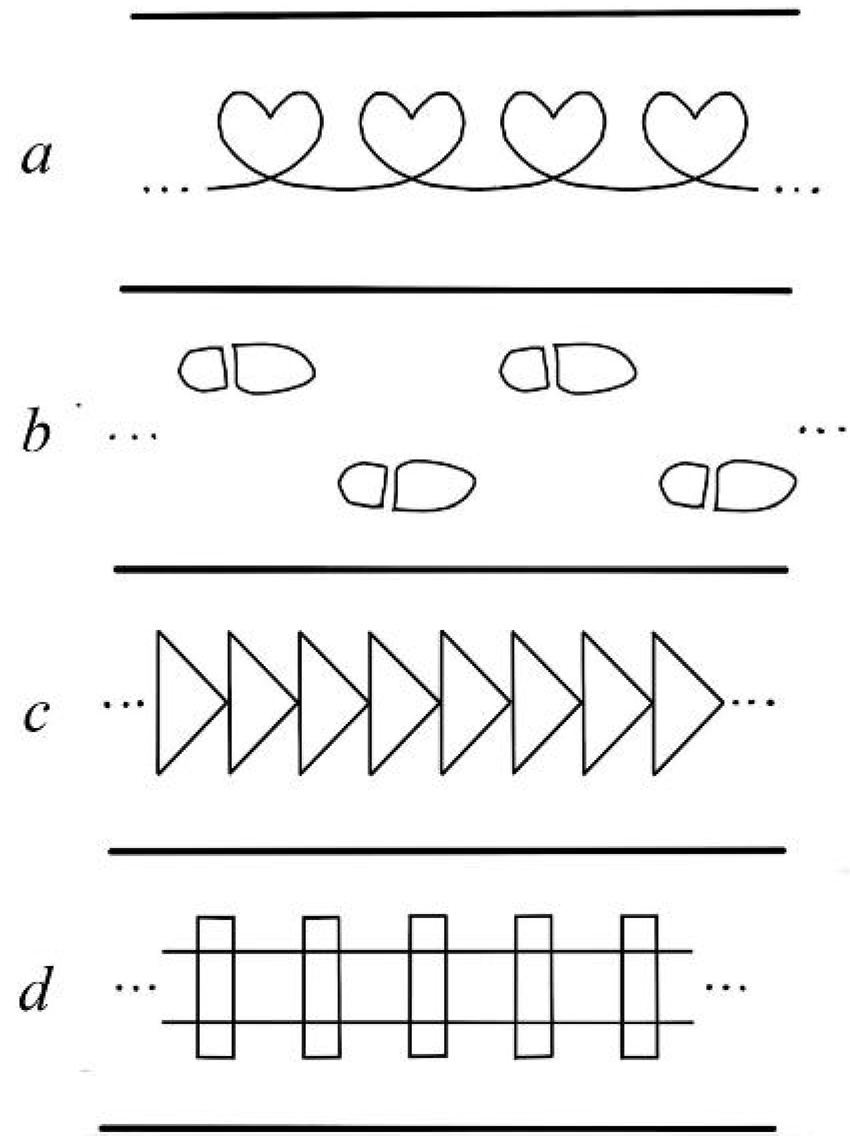


Courtesy of the The Met

Fragment from a 6th century Frieze with Acanthus-Leaf Wheels.

Summer Fun!

8. Here are the examples I came up with!



9. There are no other symmetry groups. All groups have translations. If there are no glide reflections in the group, there cannot be a horizontal reflection (since a horizontal reflection + translation = glide reflection), so the only options are 1) just rotations, 2) just vertical reflections, or 3) neither (having both would give a horizontal reflection). If there is a glide reflection and a horizontal reflection, the options are 4) just those or 5) also vertical reflections (because glides + vertical reflection gives rotation). If there is a glide reflection and no horizontal reflection, again we can have 6) just those or 7) also vertical reflections.

Summer Fun!

Calendar

Session 39: (all dates in 2026)

September	10	Start of the thirty-ninth session!
	17	
	24	
October	1	Avery Vilandrie, Shiftsmart
	8	
	15	
	22	
	29	
November	5	
	12	
	19	
	26	Thanksgiving - No meet
December	3	

Girls' Angle has run nearly 200 Math Collaborations. Math Collaborations are fun, fully collaborative, math events that can be adapted to a variety of group sizes and skill levels. We now have versions where all can participate remotely. We have now run four such "all-virtual" Math Collaborations. If interested, contact us at girlsangle@gmail.com. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

Author Index to Volume 19

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Girls' Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ______________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

The $50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- ☐ Enclosed is a check for $50 for a 1-year Girls' Angle Membership.
- ☐ I am making a tax-free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

Girls' Angle

A Math Club for Girls

Girls' Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is $20/meet for members and $30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle is based in Cambridge, Massachusetts. For security reasons, only members and their parents/guardians will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, founder and director of the Exploratory
Yaim Cooper, Institute for Advanced Study
Julia Elisenda Grigsby, professor of mathematics, Boston College
Kay Kirkpatrick, associate professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, assistant dean and director teaching & learning, Stanford University
Lauren McGough, postdoctoral fellow, University of Chicago
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls
Elissa Ozanne, associate professor, University of Utah School of Medicine
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Liz Simon, graduate student, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, associate professor, University of Washington
Karen Willcox, Director, Oden Institute for Computational Engineering and Sciences, UT Austin
Lauren Williams, professor of mathematics, Harvard University

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high-level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: _______________________

(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is $216 for one session (12 meets)	☐ I will pay on a per meet basis at $30/meet.
☐ I will pay on a per meet basis at $20/meet.	☐ I'm including $50 to become a member, and I have selected an item from the left.

☐ I am making a tax-free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: ___________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____