Girls' Angle Bulletin

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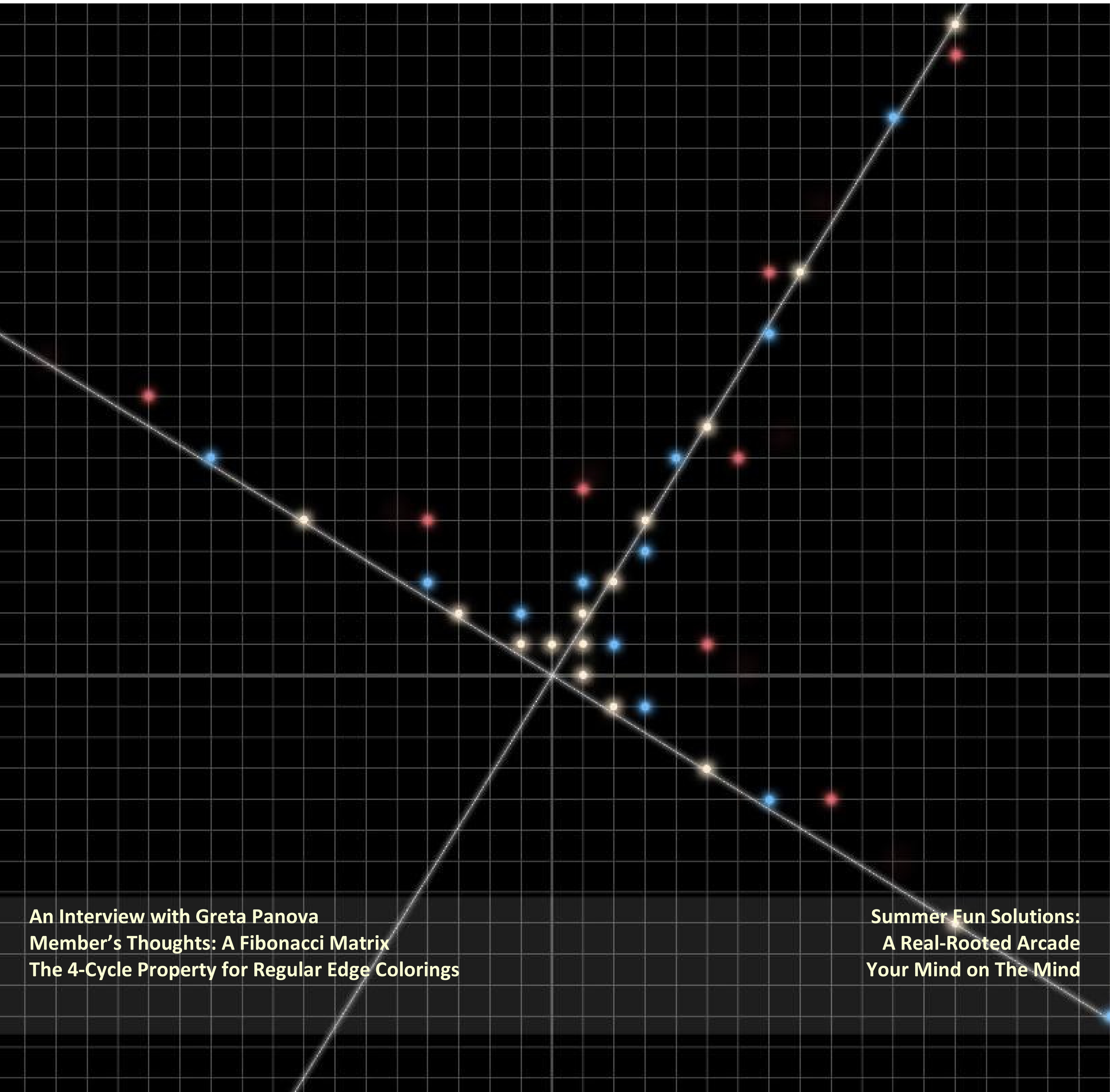
To Foster and Nurture Girls' Interest in Mathematics



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The 4-Cycle Property for Regular Edge Colorings

Summer Fun Solutions:
A Real-Rooted Arcade
Your Mind on The Mind

From the Founder

A great virtue of mathematics is that it empowers us to build glorious structures out of nothing but our own thoughts. I especially hope you'll be inspired to create some math after reading about Caitlin Cunjak's math-making adventure in Fibonacci land. -Ken Fan, President and Founder

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Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: *Golden Dagger*, by C. Kenneth Fan. Based on an idea of Caitlin Cunjak, the points (x, y) where x and y are consecutive terms in the Fibonacci (yellow) or Lucas (blue) sequences.

An Interview with Greta Panova

Greta Panova is the Distinguished Professor in Science and Engineering and Professor of Mathematics at the University of Southern California, Los Angeles. Greta earned her doctoral degree in mathematics from Harvard University under the supervision of Richard Stanley. She also holds a Master's degree in Mathematics from the University of California, Berkeley and two Bachelor of Science degrees, one in Mathematics and one in Electrical Science and Engineering, both from the Massachusetts Institute of Technology.

This interview was conducted by Elsa Frankel of Wellesley College.

Elsa: When did you first develop an interest in mathematics, and what were some of the topics and ideas that caught your attention at the time?

Greta: Sometime around 4$^{\text{th}}$ grade I realized that math was interesting and I was relatively good at it. This might have been partially because that year I was in school in Germany, in a completely new environment and new language; mathematics was the only universal tool I could communicate with. Later in school I developed a particular appreciation for geometry (Euclidean) as I really liked visualization and proofs via "additional constructions." Then I had a great opportunity to study at the National High School of Mathematics and Natural Sciences in Bulgaria, where we had a lot of extracurricular math seminars and got to see math well beyond the standard school curriculum.

Elsa: What's a favorite geometry theorem that you recall fondly from grade school?

> *Many research mathematicians did not do math competitions, because their strengths are depth, focus, and persistence which are necessary in research but not that much in competitions.*

Part of an image by Greta Panova

Greta: It's been a while, but here are a few interesting theorems: Ptolemy's theorem which says that the product of diagonal lengths of an inscribed quadrilateral is equal to the sum of products of opposite sides. Another remarkable one is Pappus' theorem which is about collinearity of points, or the Nine-point circle which says that 9 special points in a triangle lie on the same circle.

Elsa: How did you become interested in algebraic combinatorics, and how would you describe the field to a broader audience?

Greta: This happened at a relatively late stage, in my 3$^{\text{rd}}$ year of graduate school I took the "Combinatorial theory" class with Richard Stanley and really enjoyed the combination of creative constructions and tricks that come from combinatorics with the structure of algebra to guide them. This is also how I would describe the field – studying discrete objects originating from or motivated by algebra (algebraic geometry, representation theory) through purely combinatorial methods, and vice versa.

Elsa: What problems in combinatorics are you currently excited about, and why?

Greta: Currently I think a lot about "what can be solved and how," which we formalize via computational complexity that I will explain later. I also enjoy problems in discrete probability, which usually involve inequalities or asymptotics of quantities of discrete combinatorial origin often inspired by algebra.

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Thank you and best wishes,
Ken Fan
President and Founder
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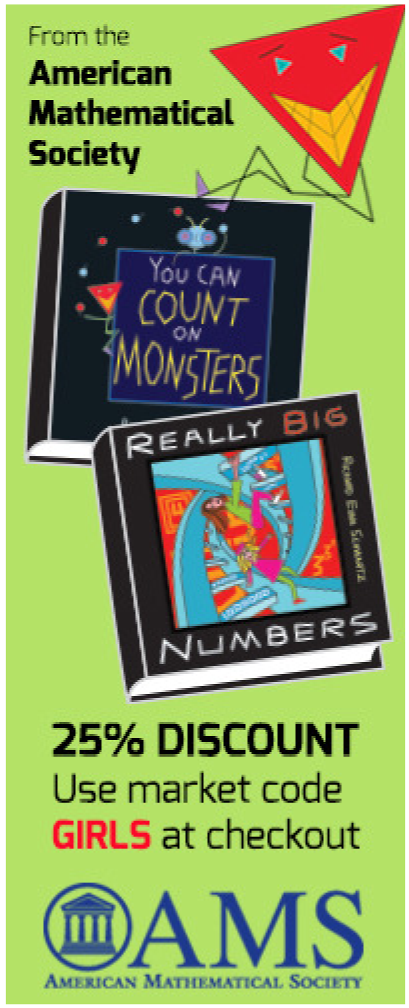

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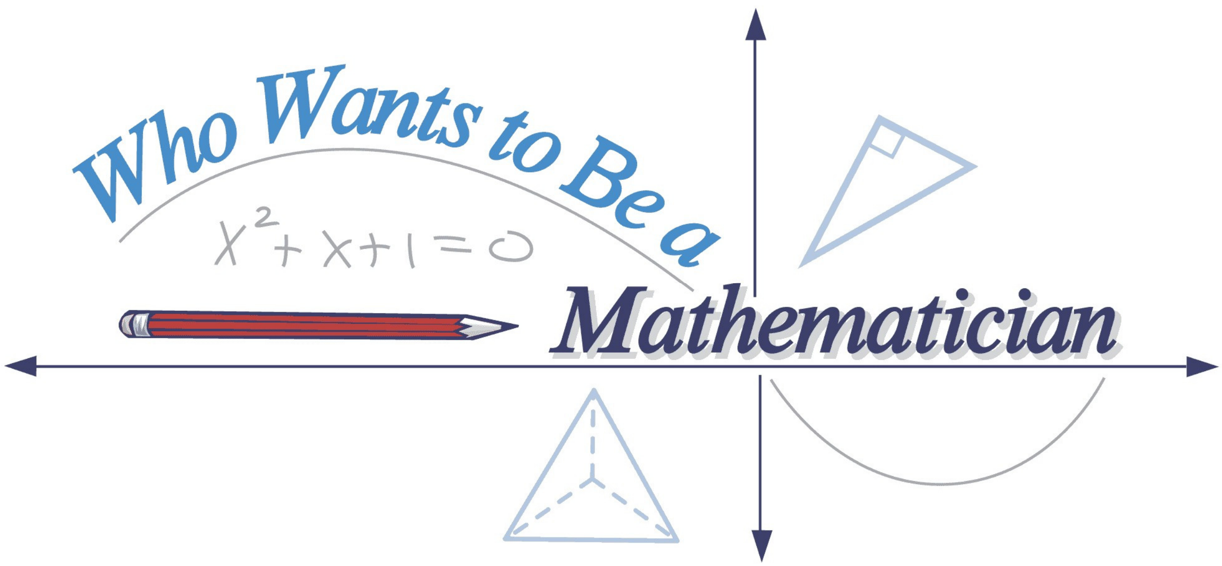


Member's Thoughts

A Fibonacci Matrix

by Ken Fan | edited by Amanda Galtman

We describe member Caitlin Cunjak's adventure with Fibonacci numbers. Caitlin is a rising 11th grader at Newton Country Day School.

This summer, Girls' Angle member Caitlin Cunjak embarked on an amazing mathematical journey that briefly ventured into Fibonacci land. Her attitude exemplifies the creative mathematical mindset. Let's retrace her path to see how she created mathematics.

We'll join her adventure in the middle, when she began to look for a formula for the nth Fibonacci number. By definition, the first two Fibonacci numbers are both 1 and each successive Fibonacci number is the sum of the previous two:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, …

More precisely, for positive integers n, we define a sequence F_n by: $F_1 = F_2 = 1$ and the recurrence relation $F_{n+1} = F_n + F_{n-1}$ for $n > 1$.

The way these Fibonacci numbers are defined, to compute, say, the 100th Fibonacci number, we would have to use the recurrence relation over and over, extending the sequence above until we reached its 100th term. Tedious!

Naturally, Caitlin wondered: **Is there a formula for the *n*th Fibonacci number?**

There is a formula. Perhaps you know it? But Caitlin had never learned it, and it's a fine question to think about, so instead of looking it up, she set off to find it herself. Besides, while everybody may be hiking toward the same mountain peak, no two paths taken to get there will be identical.

Her first idea: The Fibonacci recurrence relation, $F_{n+1} = F_n + F_{n-1}$, relates three terms of the Fibonacci sequence. If I can find two other equations involving the same three terms, I will be able to solve for the terms. Perhaps I can find a formula that way!

She began playing with triplets of consecutive Fibonacci numbers to see if she could find other equations that they satisfy. She soon noticed that F_n^2 always seemed to differ from $F_{n-1}F_{n+1}$ by 1. For example, 3, 5, and 8 are consecutive Fibonacci numbers and $5^2 = 3 \times 8 + 1$. Also, 5, 8, and 13 are consecutive Fibonacci numbers and $8^2 = 5 \times 13 - 1$. This led her to conjecture that

$$F_n^2 = F_{n-1}F_{n+1} - (-1)^n \text{ for } n > 1.$$

Instead of trying to prove this, she continued to seek out more relations among Fibonacci numbers. In the above equation, the square of a Fibonacci number is compared to the product of the two Fibonacci numbers that flank it in the sequence. This made her wonder how F_n^2 might relate to the product of the two Fibonacci numbers one term further away on each side. That is, how does F_n^2 compare to $F_{n-2}F_{n+2}$?

This led to her second conjecture:

$$F_n^2 = F_{n-2}F_{n+2} + (-1)^n \text{ for } n > 2.$$

For example, the terms near the 8th term, 21, in the Fibonacci sequence go 8, 13, 21, 34, 55. Now $21^2 = 441$ and $8 \times 55 = 440$, so $21^2 = 8 \times 55 + 1$, as her conjecture predicts.

Naturally, she decided to compare F_n^2 to $F_{n-3}F_{n+3}$ … could it be that $F_n^2 = F_{n-3}F_{n+3} - (-1)^n$?

She computed a few examples:

$$\begin{aligned} 3^2 &= 1 \times 13 - 4 \\ 5^2 &= 1 \times 21 + 4 \\ 8^2 &= 2 \times 34 - 4 \\ 13^2 &= 3 \times 55 + 4 \end{aligned}$$

Curious...

Maybe it's too bad that it was no longer +1 or -1, but it was still nice that the difference seemed to be a constant. It would be really neat, she thought, if $|F_n^2 - F_{n-k}F_{n+k}|$ depended only on k and not on n. She computed more examples:

k	1	2	3	4	5	6	7
$F_8^2 - F_{8-k}F_{8+k}$	-1	1	-4	9	-25	64	-169
$F_9^2 - F_{9-k}F_{9+k}$	1	-1	4	-9	25	-64	169
$F_{10}^2 - F_{10-k}F_{10+k}$	-1	1	-4	9	-25	64	-169
$F_{11}^2 - F_{11-k}F_{11+k}$	1	-1	4	-9	25	-64	169
$F_{12}^2 - F_{12-k}F_{12+k}$	-1	1	-4	9	-25	64	-169

Do you see what Caitlin saw?

Not only are the absolute values along each column the same, but they are also perfect squares!

And what are the square roots of the absolute values of these entries?

The Fibonacci numbers!

And so we arrive at Caitlin's next conjecture, which subsumes the first two:

$$F_n^2 = F_{n-k}F_{n+k} + (-1)^{n+k}F_k^2 \text{ for } n > k.$$

Exciting!

But is it true?

It is; Caitlin managed to prove it.

In the process of proving it, Caitlin noticed that one can drop the condition that $n > k$ if one extends the Fibonacci sequence backwards by insisting that the recurrence relation hold:

…, 34, -21, 13, -8, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, …

That is, we extend F_n to $n \leq 0$ by requiring the recurrence relation $F_{n+1} = F_n + F_{n-1}$ to hold for all n. Notice that $F_{-n} = (-1)^{n+1}F_n$.

There are many ways to prove that $F_n^2 = F_{n-k}F_{n+k} + (-1)^{n+k}F_k^2$, so rather than show you Caitlin's proof, we urge you to dream up your own.

It turns out that this nifty identity was already noted by Catalan, who recorded it in one of his notebooks dated October, 1879. Today, it is known as Catalan's identity.

Caitlin also rediscovered Vajda's generalization of Catalan's identity, which says

$$F_{n+a}F_{n+b} - F_nF_{n+a+b} = (-1)^n F_a F_b.$$

Despite rediscovering these pretty identities relating Fibonacci numbers, Caitlin was still unable to find a formula for the nth Fibonacci number.

What to do?

Caitlin: **Maybe I can't find a formula for the nth Fibonacci number, but perhaps I can solve a closely related problem. Suppose A and B are terms of a sequence that satisfies the Fibonacci recurrence relation and are separated by N terms. Can I find a formula for the terms between A and B?**

What a refreshing tweak of the original question!

To answer this, Caitlin decided to tackle the $N = 3$ case first. When $N = 3$, we have five numbers,

$$A, x, y, z, B,$$

that satisfy the Fibonacci recurrence relation. Therefore,

$$\begin{aligned} y &= x + A \\ z &= y + x \\ B &= z + y \end{aligned}$$

A system of 3 linear equations in 3 unknowns! We can rearrange the terms so that the unknowns appear only to the left of the equal signs:

$$\begin{aligned} x - y &= -A \\ x + y - z &= 0 \\ y + z &= B \end{aligned}$$

In matrix form, these equations can be written

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -A \\ 0 \\ B \end{pmatrix}.$$

Using whichever technique you prefer to find the inverse of the matrix, you'll find that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} -A \\ 0 \\ B \end{pmatrix}.$$

That is,

$$\begin{aligned} x &= (-2A + B)/3, \\ y &= (A + B)/3, \\ z &= (-A + 2B)/3. \end{aligned}$$

For $N = 4$, with the terms of the sequence labeled A, x, y, z, w, B, Caitlin found

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -A \\ 0 \\ 0 \\ B \end{pmatrix} \text{ and } \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 & 2 & 1 & 1 \\ -2 & 2 & 1 & 1 \\ 1 & -1 & 2 & 2 \\ -1 & 1 & -2 & 3 \end{pmatrix}\begin{pmatrix} -A \\ 0 \\ 0 \\ B \end{pmatrix}.$$

What patterns do you see?

For general N, if we label the terms of the sequence A, x_1, x_2, x_3, …, x_N, B, Caitlin found

$$\begin{pmatrix} 1 & -1 & 0 & 0 & \dots & 0 \\ 1 & 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & 1 & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 1 & -1 \\ 0 & 0 & \dots & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{N-1} \\ x_N \end{pmatrix} = \begin{pmatrix} -A \\ 0 \\ 0 \\ \vdots \\ 0 \\ B \end{pmatrix}.$$

The N-by-N coefficient matrix, which we'll denote by M_N, has 1s along its main diagonal and on the diagonal just below the main diagonal, -1s along the diagonal just above the main diagonal, and 0s everywhere else.

Using a combination of pattern observation and computation, Caitlin guessed the inverse of M_N. For example, in the inverse of M_N, it seems that the last column appears to be the first N terms of the Fibonacci sequence. The first column appears to be, up to sign, the N terms of the Fibonacci sequence starting from term $-N$. (Recall that Caitlin extended the Fibonacci sequence to negative indices.) And the determinant of M_N … that turns out to be F_{n+1}! Isn't that neat? After guessing the explicit form of the inverse of M_N and its determinant, Caitlin proved that her guess was correct.

Unfortunately, the fact that the determinant of M_N is F_{N+1} has already been published. It appears as Exercise 24 at the end of Section 1.2.8 of Volume 1 of Donald Knuth's *The Art of Computer Programming*, which was first published in 1968 … long before Caitlin was born, but almost 100 years after Catalan recorded his identity. Even so, it's pretty cool that she rediscovered a mathematical fact that someone as distinguished as Donald Knuth saw fit to publish.

A consequence of Caitlin's determination of the inverse of M_N is the formula

$$x_k = \frac{BF_k + (-1)^N AF_{-N-1+k}}{F_{N+1}}.$$

It's often fruitful to think about whether one might have been able to deduce such a nice formula directly. This formula shows that x_k is a linear combination of Fibonacci sequences, specifically, it is B/F_{N+1} times the Fibonacci sequence, plus $(-1)^N A/F_{N+1}$ times the Fibonacci sequence shifted to the right by $N + 1$. Linear combinations of sequences that satisfy the Fibonacci recurrence relation must also satisfy the Fibonacci recurrence relation. Therefore, the explicit representation of x_k as just such a linear combination guarantees that $x_1, \ldots, x_N$ interpolate between A and B in such a way that the result satisfies the Fibonacci recurrence relation. But why this particular linear combination?

Let's look at the sequences F_k/F_{N+1} and $(-1)^N F_{-N-1+k}/F_{N+1}$ for several k values:

k	**0**	**1**	**2**	**3**	**…**	**N**	**$N + 1$**
F_k/F_{N+1}	0	$1/F_{N+1}$	$1/F_{N+1}$	$2/F_{N+1}$	**…**	F_N/F_{N+1}	1
$(-1)^N F_{-N-1+k}/F_{N+1}$	1	$(-1)^N F_{-N}/F_{N+1}$	$(-1)^N F_{-N+1}/F_{N+1}$	$(-1)^N F_{-N+2}/F_{N+1}$	**…**	$(-1)^N/F_{N+1}$	0

Notice the entries in the 0 and $N + 1$ columns! By adding A times the second sequence ($(-1)^N F_{-N-1+k}/F_{N+1}$) to B times the first sequence (F_k/F_{N+1}), we obtain a sequence that satisfies the Fibonacci recurrence relation AND has its zeroeth term equal to A and its $(N + 1)$th term equal to B!

Do you see how these observations provide a method for interpolating between two given values to create a sequence that satisfies *any* linear recurrence relation at all, not just the Fibonacci one?

Also, can you deduce a criterion on the values of A and B that tells when the interpolated values will be integers?

The Journey Continues

The above method does more than interpolate *between* two given values A and B. By letting k be any integer, we automatically get the continuations of the sequence preceding A and going beyond B.

Notice that terms 1 and 2 of the sequence F_{k-1} are 0 and 1, respectively, and terms 1 and 2 of the sequence F_{k-2} are 1 and 0, respectively. Using the same technique of taking linear combinations of sequences that satisfy the Fibonacci recurrence relation, we see that the Fibonacci-like sequence that begins A, B, … must be $AF_{k-2} + BF_{k-1}$.

All this beautiful math, yet a formula for the nth Fibonacci number remains elusive. Can linear combinations of sequences that obey the Fibonacci recurrence relation be used to obtain an explicit formula? If so, it seems that the linear combinations would have to be built from sequences that satisfy the Fibonacci recurrence relation *and for which we do have an explicit formula for the* n*th term*.

Caitlin: **Can we find any sequence that has an explicit formula and satisfies the Fibonacci recurrence relation?**

For what sequences do we have explicit formulas for the nth term? Perhaps arithmetic and geometric sequences come to mind?

But if we check to see which arithmetic sequences satisfy the Fibonacci recursion relation, we find that there is only one: the sequence whose every term is zero – not helpful for our purposes.

What about geometric sequences? Since scaling a sequence doesn't change its adherence to the recurrence relation, we can assume that 1 is one of the terms of the sequence. If the common ratio is r, then the sequence would include the terms

$$\ldots, 1/r^3, 1/r^2, 1/r, 1, r, r^2, r^3, \ldots.$$

To satisfy the Fibonacci recurrence relation, we need $r^{n+1} = r^n + r^{n-1}$ or, dividing by r^{n-1} throughout, $r^2 = r + 1$. A quadratic equation! Its roots are $r_+ = \frac{1+\sqrt{5}}{2}$ and $r_- = \frac{1-\sqrt{5}}{2}$. So, up to a constant multiple, there are *two* geometric sequences that satisfy the Fibonacci recurrence relation. One goes

$$\ldots, 1, r_+, r_+^2, r_+^3, r_+^4, \ldots$$

and the other goes

$$\ldots, 1, r_-, r_-^2, r_-^3, r_-^4, \ldots.$$

Armed with these two Fibonacci-recurrence-relation-satisfying sequences for-which-we-know-an-explicit-formula-for-the-nth-term, we can apply the linear combination technique and know that the sequence $Ar_+^n + Br_-^n$ must also satisfy the Fibonacci recurrence relation. If we choose A and B so that $Ar_+^0 + Br_-^0 = 0$ and $Ar_+^1 + Br_-^1 = 1$, we will get the desired explicit formula for the Fibonacci numbers!

Solving the system of linear equations in A and B, we find:

$$A = 1/\sqrt{5} \text{ and } B = -1/\sqrt{5}.$$

Thus, Caitlin rediscovered the Binet formula for the nth Fibonacci number:

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

What a journey!

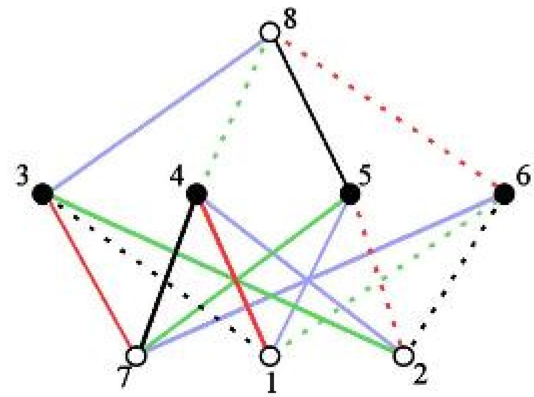
The 4-Cycle Property for Regular Edge Colorings[1]

by Robert Donley[2]
edited by Amanda Galtman

Before engaging with the main themes of this installment in the series, we note two real-world applications of regular edge colorings. For a tournament in which players are paired at most once, the roster describes the adjacency list of a regular edge coloring. If all pairings of players occur, we obtain a Latin square. If the players belong to two teams of the same size, then the bipartite condition applies.

In the study of a subject called supersymmetry in particle physics, bipartite regular edge colorings are used to describe configurations of particles. These particles, called **bosons** and **fermions**, correspond to the parts in the bipartition of vertices. For a given choice of edge color, if we exchange the vertices on each edge of that color, then the term “supersymmetry” refers to these “perfect” (as in “perfect matching”) exchanges of bosons and fermions.

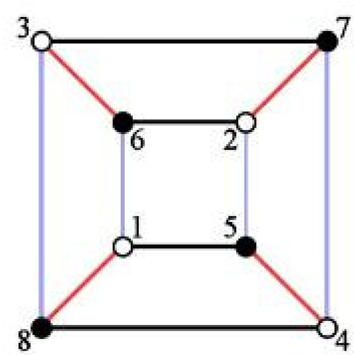
In 2005, Michael Faux and S. James Gates, Jr. introduced the Adinkra, a graphical device for solving differential equations in supersymmetry. The Adinkra definition begins with a bipartite regular edge coloring and has further sophisticated conditions. In this installment, we explore the next condition, the 4-cycle property. (We omit the remaining parts of the definition: the totally odd dashing and the compatible partial order.) On the right is a typical Adinkra graph.



Instead of a theory based on perfect matchings and permutation matrices, our new approach for creating regular edge colorings combines binary codes and Latin rectangles. It will be helpful to review the installment “Adjacency Lists and Latin Rectangles” (Volume 18, Number 2) for development of adjacency lists and the quadrilateral property, which we call the 4-cycle property in this installment. The installment “Regular Edge Colorings on Hypercubes” (Volume 18, Number 1) provides background on hypercubes and bitstrings, and it will be especially useful to refer to the big diagram for the hypercube Q_4.

To motivate the 4-cycle property, consider the regular edge coloring of the 3-cube with the following graph and adjacency list; recall that the adjacency list forms a Latin rectangle.

Q_3	1	2	3	4	5	6	7	8
Red	8	7	6	5	4	3	2	1
Blue	6	5	8	7	2	1	4	3
Black	5	6	7	8	1	2	3	4



[1] This installment is 22nd in a series that began in Volume 15, Number 3. It is also part 9 of a subseries that began in Volume 17, Number 4.
[2] This content is supported in part by a grant from MathWorks.

Since the 3-cube is bipartite, the table naturally splits, and it is sufficient to present half of the table. The boxed entries in the first two rows exhibit the adjacency list property for a regular edge coloring; if 6 is in the red row and the column for 3, then 3 is also in the red row and column for 6. These boxes always form a rectangle. The boxed entries in the lower right-hand corner reflect the 4-cycle property. We recall its definition.

Definition: A regular edge coloring satisfies the **4-cycle property** if, when the graph is restricted to edges of any two colors, the resulting set of disjoint cycles consists only of 4-cycles.

In terms of the adjacency list, if we delete the top row and all but two other rows, then for any columns in the remaining array, there is another column with the same entries but inverted. These entries always form a rectangle. If we delete the top two rows in the 3-cube example, the boxed entries refer to the black-blue 4-cycle with vertices 3, 7, 4, and 8.

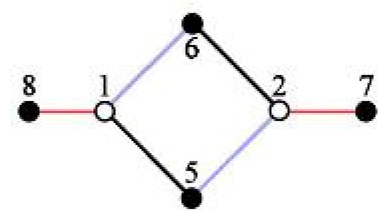
Exercise: In the 3-cube example, verify the adjacency list property for each pair of vertices, and verify the 4-cycle property by eliminating rows and identifying bicolor 4-cycles in the graph.

Exercise: For the Adinkra graph above, determine the adjacency list, verify the adjacency list property, and verify the 4-cycle property. What familiar graph is this?

For an applied interpretation of the 4-cycle property, let's consider walks in the graph.

Definition: A **walk** in a graph is a single vertex or a sequence of consecutive adjacent vertices. A **path** is a walk in which no vertices repeat.

We denote a walk by a list of vertices with dashes. For instance, with the 4-cycle property, the walk 8-1-6-2-7 on the right (which shows part of the 3-cube above) can be replaced by the walk 8-1-5-2-7 without changing the number of black and blue edges.



Let's also consider a walk as a **color sequence**, read from left to right. With the starting vertex 8, the walk 8-1-6-2-7 may be represented by the color sequence Red, Blue, Black, Red; edge color regularity assigns each item in the sequence to a new vertex. On the other hand, the walk 8-1-5-2-7 is given by the sequence Red, Black, Blue, Red. The operation of changing the walk to the other side of the bicolor 4-cycle is just the exchange of the two colors in the color sequence.

With the 4-cycle property, if we have a walk determined by a color sequence, possibly with multiple instances of a color, any permutation of the colors results in another walk with the same initial and final vertices. Similarly, removing an adjacent pair of identical colors from such a sequence gives a walk with the same initial and final vertices. For instance, the sequence Red, Red, Red, Blue for the walk 8-1-8-1-6 may be shortened to the list Red, Blue, which gives the walk 8-1-6. Together, these operations on walks give the following lemma.

Walk Reduction Lemma: Suppose we have a regular edge coloring with s colors $c_1, \ldots, c_s$ and with the 4-cycle property. Let $w = c_{i_1} \ldots c_{i_k}$ be a walk between vertices v_1 and v_2. Then there exists another walk w' with the same initial and final vertices such that each color appears at most once. Call such a walk **reduced.**

Exercise: Prove the Walk Reduction Lemma. Is the reduced walk unique? If not, how many reduced walks are associated to a given walk w?

Exercise: In the 3-cube above, determine all paths from 8 to 7; each vertex should be used at most once. Use your proof of the Walk Reduction Lemma to reduce each path to 8-3-7.

If we fix an ordering on the colors, then the result of the Walk Reduction Lemma is uniquely determined.

Exercise: For 3-cube, fix the ordering Red, Blue, Black. How many ordered reduced color sequences exist, including the empty sequence? List these, and, starting from the initial vertex 8, determine the final vertex for each path.

Exercise: In the previous exercise, associate to each sequence a bitstring with entries of 1 in the places where a color appears in the ordered sequence and 0 otherwise. Label each endpoint with the bitstring, and verify that vertices joined by an edge differ by 1 in exactly one entry.

Exercise: Repeat the previous two exercises for the hypercube Q_4 with the ordering Black, Blue, Red, Green. The graph appears in "Regular Edge Colorings on Hypercubes" (Volume 18, Number 1). Remove the bitstring labels on the vertices, rename them with integers, and recover the original bitstrings using the color sequences. The vertex labeled with the bitstring 0000 should be the initial vertex when applying color sequences.

For the general hypercube Q_s with the parallel edge coloring, the path model with color sequences extends naturally. In fact, for a regular edge coloring of a connected graph with the 4-cycle property, this path model for hypercubes still applies, but an ordered reduced walk may return to the initial vertex early, forming a cycle. In this case, each vertex is obtained from a reduced color sequence, but we obtain paths by removing any cycles from the walk list or color sequence.

All this reasoning leads to the striking result that a connected graph with a regular edge coloring and the 4-cycle property is a quotient of a hypercube by a **linear binary code**, defined below. Since our interest lies in developing new examples of regular edge colorings, we instead explore binary codes and show how they work in practice with examples.

Definition: A **binary code** is a collection of bitstrings, possibly of different lengths. An element of a binary code is called a **codeword**. A **binary block code** is a binary code in which all codewords have the same length, called the **block length**. Such a code is called **linear** if, for any two codewords, their sum under binary addition is also a codeword. (Such codes are used in communications.)

As we have seen above, the vertices of the hypercube Q_n give the binary block code of all bitstrings of length n. In this work, we assume all binary codes are binary block codes.

Definition: A codeword is called **even** if it contains an even number of 1s, and **odd** otherwise. A code is called **even** if every codeword is even.

Exercise: Prove that the sum of two even or two odd codewords is even. Prove that the sum of an odd and an even codeword is odd.

By the previous exercise, the subset of even bitstrings forms a linear code.

Definition: Let S be a set of bitstrings. The code $C(S)$ **generated** by S is the set of all sums of bitstrings in S. If each element of a code C can be written uniquely as sum of elements of S, then we call S a **basis** for C.

Exercise: Prove that the set E of even codewords of length 5 is generated by

$$S = \{11000, 01100, 00110, 00011\}.$$

List all codewords for E, and prove that S is a basis for E.

We now show how to build a regular edge coloring on a connected graph from a linear code C. Let s be the block length and let B be a basis for C. Let k be the number of elements in B. We will construct a regular edge coloring on a connected graph with 2^{s-k} vertices and s colors. The way we do this is to start with a "base vertex" v, and build out from it subject to the existence of certain cycles specified by the codewords. To associated a cycle with a codeword, we fix an association between the colors and the digit positions. Then, for each codeword, we get a color sequence by noting the positions of the 1s. We then build out the graph insisting that these color sequences be **relations**, where a relation is a color sequence in which any path starting at v and moving along edges according to the color sequence forms a cycle. (We shall illustrate this process below.)

In general, we want restrictions on our codewords. For instance, a codeword with a single nonzero entry would correspond to a loop (one color), and a codeword with two nonzero entries would correspond to a multiple edge (two colors), so we disallow such codewords because we desire only simple graphs (i.e., graphs with no loops or multiple edges).

For constructing examples, it is enough to use relations in the following way:

Proposition: Suppose the graph is connected. If the color sequence R forms a cycle at the base vertex v, then R forms a cycle at any other vertex u. That is, every cycle corresponds to a relation, and, once its color sequence is reduced and ordered, a codeword.

Proof. Let P be a color sequence that forms a walk between v and the final vertex u, and let P' be the reverse sequence. If we form the walk associated to P' with initial vertex u, then the final vertex is v. If R is the color sequence that forms the cycle for a relation, then $P'RP$ is a loop that goes from u to v, back to v, then back to u. By the walk reduction lemma, we can reduce $P'RP$ to R which starts and ends at the same place $P'RP$ starts and ends, namely, at u. Thus, R forms a cycle at u. □

Exercise: Prove that concatenation of cycles at v corresponds to addition of codewords.

Exercise: Prove that, if the code is even, then the resulting graph is bipartite. Recall that a graph is bipartite if and only if every cycle in the graph has an even number of edges.

For a non-trivial linear code, we consider the regular edge coloring associated to

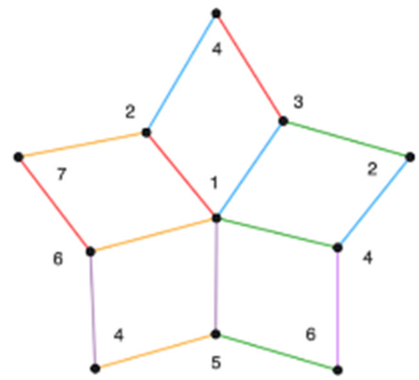
$$C_1 = \{00000, 11100, 00111, 11011\}.$$

The graph of C_1 has five colors and eight vertices, but, since it has odd codewords, it is not bipartite. Let's order the five colors as

Red (R), Blue (B), Green (G), Purple (P), Orange (O)

A triangle condition on the graph is given by the relations RBG and GPO; that is, for instance, if a red and blue edge are adjacent, then these edges form a triangle with some green edge.

To proceed, we start with vertex 1. Around vertex 1, we add an edge for each color and name the new vertices 2 through 6. Then we include the remaining edges using the 4-cycle property. By the relation RBG, vertex 4 must be at the top of the graph, and vertex 2 must be on the right. Likewise, the relation GPO also identifies the bottom vertices as 4 and 6 (which we replicate there instead of using a long curve to attach it to the already existing vertices labeled 4 and 6; we just recognize that all the vertices labeled by the same number are to be considered the same vertex). See the figure at right.



Exercise: Prove that the remaining vertex cannot be one of the vertices 1 through 6.

Exercise: Label the remaining vertex 7. Set up the adjacency list for C_1, and fill in the entries based on the above graph. The table should have five rows and eight columns.

Exercise: Fill in the remaining entries using the Latin rectangle property, the adjacency list property, and the 4-cycle property. Then draw the graph of C_1 with the regular edge coloring. The solution for the adjacency list is given at the end of this article.

We can follow color sequences directly in the adjacency list. Given an initial vertex v, we start in the v column, find the row with the first color, record the entry, and repeat the process with each new entry until the sequence is exhausted. For a relation, the last entry is v.

Let's compare how RBG and BGR form cycles at vertex 7 in the adjacency list solution at the end of this installment. For RBG applied to 7, 6 is in the red row and column 7, then 8 is in the blue row and column 6, and finally 7 is in the green row and column 8, closing the triangle 7-8-6-7. For the relation BGR, this method gives the walk 7-5-6-7.

Exercise: With the graph of C_1, verify that the 4-cycle property holds by drawing the bicolor cycles for each pair of colors. Verify that the triangle relations hold at the vertex 1 both by inspecting triangles in the graph and by tracing each relation in the adjacency list.

Exercise: Find the adjacency list and regular edge coloring for the linear code

$$C_2 = \{00000, 11110, 10101, 01011\}.$$

If you repeat the work from C_1, explain why the three unused vertices cannot be labeled 1 through 6. Then use the Latin rectangle property to place labels 7 and 8 in the partial graph.

Next, we consider the linear binary code

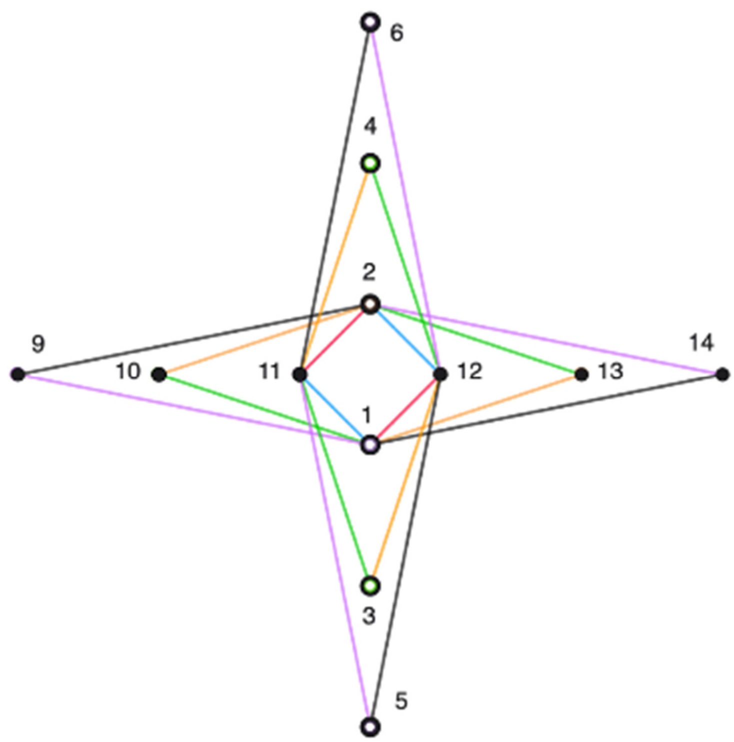
$$d_6 = \{000000, 111100, 001111, 110011\}.$$

The graph has six colors and 16 vertices. Although we have more vertices and an extra color, this graph is at the same level of difficulty as C_1. Since each codeword is even, the graph is bipartite, and the halved adjacency list has six rows and eight vertices. We order the colors as

Red (*R*), Blue (*B*), Green (*G*), Orange (*O*), Purple (*P*), Black (*K*),

and separate the vertices into parts 1 through 8 and 9 through 16.

In addition to the 4-cycle property, we also have the 4-cycle relations *RBGO*, *GOPK*, and *RBPK*. In the partial graph below, assume the vertices 1 and 2 are joined by red-blue paths. By the relations, we can extend the graph in four directions until the colors at vertices 1, 2, 11, and 12 are exhausted.



Exercise: Prove that, unlike the diagram for C_1, the vertices in this partial graph must have distinct labels.

We now have enough information to complete the regular edge coloring.

Exercise: Set up the adjacency list for d_6 and fill in the entries based on the partial graph.

In the red row, vertices 7 and 8 are missing. Likewise, the missing entries in the column for vertex 10 are 7 and 8 since the Latin rectangle property rules out 5 and 6. Vertices 7, 8, 15, and 16 are still unused, and there are several valid options to place the 7s.

Exercise: Complete the adjacency list and draw the regular edge coloring in valise form. That is, arrange the vertices of the graph along separate rows according to the bipartition.

The solution for half of the adjacency list is at the end of this article.

Exercise: One advantage of the bipartite case is that only half of the adjacency list is required to describe the graph. Determine how to adapt the process for tracing a color sequence in half of the adjacency list. Find the 4-cycle determined by the relation *BORG* starting at vertex 8.

Exercise: What happens for the linear code $C = \{00 \ldots 00, 11 \ldots 11\}$? What happens if we add the codeword $11 \ldots 11$ to C_1 or d_6?

As a final example, we consider the linear binary code e_7, which is generated by the codewords 1111000, 0011110, and 1010101. The graph for e_7 has seven colors and 16 vertices.

Exercise: Find all eight codewords for e_7. Is the graph of e_7 bipartite?

If we remove the last digit from their bitstrings, then the first two generating codewords of e_7 are the generators of d_6, so we should try to build the graph of e_7 from the graph of d_6. It turns out that this attempt succeeds.

Let the new color be Yellow (Y). The new relations, corresponding to the codeword 1010101, is *RGPY*. Since the graph of e_7 is bipartite, we need only add another row to the halved adjacency list for d_6.

Exercise: To fill in the last row of e_7, either use the relation *RGPY* on the graph of d_6 or trace this relation in the adjacency list. Then verify that the 4-cycle property holds with yellow edges. Finally, draw the graph of e_7 in valise form by adding the yellow edges to the graph of d_6.

As before, the solution for the last row is given below.

Solutions:

C_1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Red	2	1	4	3	8	7	6	5
Blue	3	4	1	2	7	8	5	6
Green	4	3	2	1	6	5	8	7
Purple	5	8	7	6	1	4	3	2
Orange	6	7	8	5	4	1	2	3

C_2	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Red	2	1	7	6	8	4	3	5
Blue	3	7	1	8	6	5	2	4
Green	4	6	8	1	7	2	5	3
Purple	5	8	6	7	1	3	4	2
Orange	6	4	5	2	3	1	8	7

d_6	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**
Red	6	4	2	1	3	5	7	8
Blue	5	3	1	2	4	6	8	7
Green	7	1	3	4	2	8	6	5
Orange	8	2	4	3	1	7	5	6
Purple	1	7	5	6	8	2	4	3
Black	2	8	6	5	7	1	3	4

e_7	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**
Yellow	4	6	8	7	5	3	1	2

Summer Fun!

In the previous issue, we presented the 2025 Summer Fun problem sets.

In this issue, we give solutions to many of the problems. Our solutions may be terse and, in some cases, are more of a hint than a solution. We prefer not to give detailed solutions before we know that most of the members have spent time thinking about the problems. The reason is that *doing* mathematics is very important if you want to learn mathematics well. If you haven't tried to solve these problems yourself, you won't gain as much when you read these solutions.

If you haven't thought about the problems, we urge you to do so *before* reading the solutions. Even if you cannot solve a problem, you will benefit from trying. By working on the problem, you will force yourself to think about the associated ideas. You will gain familiarity with the related concepts and that will make it easier and more meaningful to read other's solutions.

With mathematics, don't be passive! Be active!

Move your pencil and move your mind – you might discover something new.

Also, the solutions presented are *not* definitive. Try to improve them or find different solutions.

Solutions that are especially terse will be indicated in red. Please do not get frustrated if you read a solution and have difficulty understanding it. If you run into difficulties, we are here to help! Just ask!

Members: *Don't forget that you are more than welcome to email us with your questions and solutions!*

Please refer to the previous issue for the problems.

Summer Fun!

A Real-Rooted Arcade

by Elisabeth Bullock | edited by Jennifer Sidney

1. We have

$$f(t) + g(t) = 2t^3 - 12t^2 + 22t - 12 = 2(t - 1)(t - 2)(t - 3),$$

so the claw machine should pass over the teddy bear three times as intended. The nth time the claw passes over the bear under the instructions $f(t) + g(t)$ is between the nth times for $f(t)$ and $g(t)$.

2. The same pattern should hold as in Problem 1. The claw machine will still pass over the bear three times, at times in between those for f and g.

3. Let the roots of f be $f_1 < f_2 < f_3$ and the roots of g be $g_1 < g_2 < g_3$. Note that these roots interlace: $f_1 < g_1 < f_2 < g_2 < f_3 < g_3$. Following the pattern from Problems 1 and 2, we want to show that $h(t) = af(t) + bg(t)$ has roots $h_1 < h_2 < h_3$ with $f_i < h_i < g_i$. Now,

$$h(f_1) = af(f_1) + bg(f_1) = bg(f_1) < 0,$$

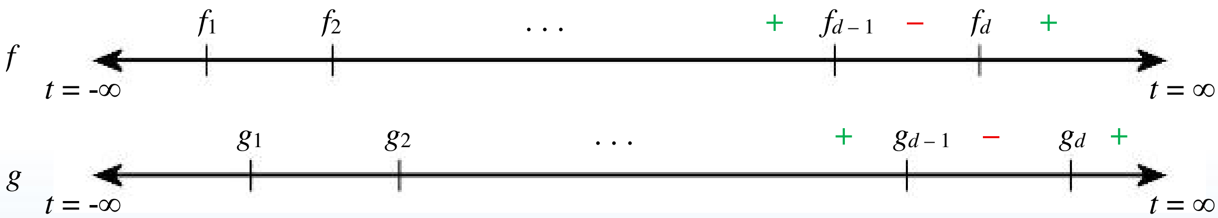
since $g_1 > f_1$. On the other hand,

$$h(g_1) = af(g_1) + bg(g_1) = af(g_1) > 0.$$

Using the tool below the problem statement, we see that because h switches signs between f_1 and g_1, it must have a zero h_1 in the interval (f_1, g_1). This argument can be repeated for the other intervals (f_i, g_i) to show that $h(f_2) > 0 > h(g_2)$ and $h(f_3) < 0 < h(g_3)$.

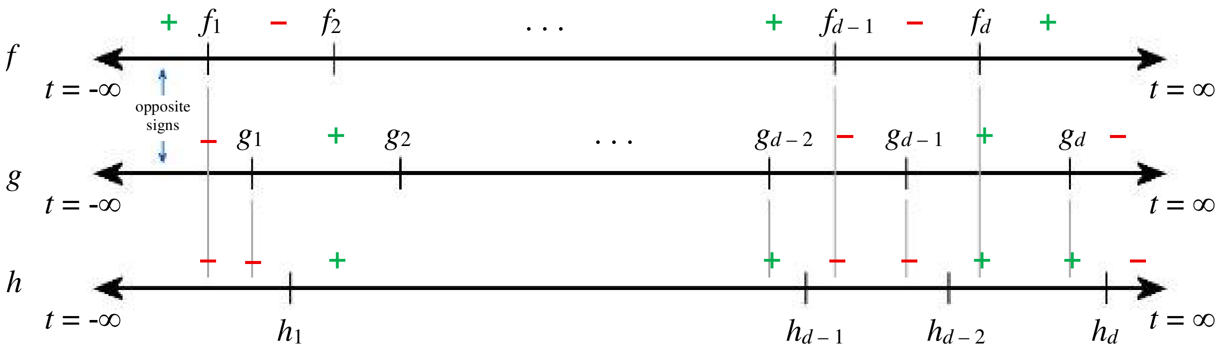
4. Let $h = f + g$. The proof is essentially the same as the one in Problem 3, but we need to be a little careful about signs. We will do the case that f and g have the same degree, as the case where the degree of f is one less than the degree of g is essentially the same.

First, we assume that the leading coefficients of f and g have the same sign, which we can assume to be positive. For simplicity, let $f_0 = g_0 = -\infty$ and let $f_{d+1} = g_{d+1} = \infty$. Then f is positive on the intervals of the form (f_{d-2i}, f_{d-2i+1}) for integers $0 \le i \le d/2$, and similarly for g. This is represented by the picture below:



Following the pattern from the previous problems, we want to show that there is a root h_j of h between f_j and g_j for each $j = 1, \ldots, d$. If $j = d - 2i$, then $h(f_j) = g(f_j) \leq 0$, since $g_{j-1} \leq f_j \leq g_j$. Also, $h(g_j) = f(g_j) \geq 0$, since $f_j \leq g_j \leq f_{j+1}$. Thus, $h(h_j) = 0$ for some $f_j \leq h_j \leq g_j$. If $j = d - 2i + 1$, we can apply the same argument to see that $h(f_j) \geq 0 \geq h(g_j)$. In summary, we have $f \ll h \ll g$.

Next, we assume the leading coefficients of f and g have opposite signs. By symmetry, we may assume that the leading coefficient of f is positive and the leading coefficient of g is negative. Further, let's assume that $f + g$ is degree d. This can be pictured as below in the subcase for which d is even and the leading coefficient of h is negative:



In this subcase, we find that $h(f_{d-2i}) \geq 0 \geq h(g_{d-2i-1})$ and $h(f_{d-2i+1}) \leq 0 \leq h(g_{d-2i})$. Thus, h has a root h_j between g_j and f_{j+1} for all $j = 1, \ldots, d - 1$. By the fundamental theorem of algebra, we have one more root, which we'll call h_d. We can appeal to the fact that complex roots come in conjugate pairs in order to deduce that this last root must be real, but we also want to know something about its location. Since the leading coefficient of h is negative, we must have $h_d \geq h_{d-1}$ in order for $h(t)$ to go to $-\infty$ as t goes to ∞, since the sign of h will alternate between roots. Further, in this case we must have $h_d \geq g_d$, since $h(g_d) \geq 0$. Thus $h \gg f, g$. If the leading coefficient of h is positive, then the end behavior of h matches that of f (i.e., both $h(t)$ and $f(t)$ go to ∞ or they both go to $-\infty$, as t goes to $-\infty$). Thus $h_d \leq f_1$, since $h(f_1)$ has the opposite sign of what we want for the end behavior of h; so in this case, $h \ll f, g$. A final case to consider is when $f + g$ is degree $d - 1$ (i.e., the leading terms cancel). By the same argument as above, we have roots h_i of h satisfying $g_i \leq h_i \leq f_{i+1}$ for all $i = 1, \ldots, d - 1$, i.e., $h \ll f, g$.

5. With just this information, there is no restriction on how many real roots $f + g$ might have. For example, consider $f(x) = x^{2d}$ and $g(x) = (x - 1)^{2d}$, which have roots 0 and 1 with multiplicity $2d$, respectively. In this example, $f(x) + g(x)$ has no real roots, since $x^{2d} + (x - 1)^{2d} > 0$ for all real x. On the other hand, consider $f(x) = (x + 1)(x + 2) \cdots (x + d)$ and $g(x) = cx^{2d}$ for some constant c. If we make c sufficiently small, $g(x)$ is very small for all $-d \leq x \leq d$, so $f(x) + g(x)$ will still have d roots, very close to -1, -2, …, $-d$. There are many more examples!

Summer Fun!

6. First we expand the a_i's in terms of the r_i's. A contribution towards the coefficient of x^i means we choose i factors of the form $(x - r_j)$ to contribute an x, and the remaining $d - i$ factors contribute their $-r_j$. Thus $a_i = (-1)^{d-i} \sum_{j_1<\ldots<j_{d-i}} r_{j_1} \cdots r_{j_{d-i}}$.

Similarly, we can compute the coefficient of x^{i-1} in $\sum_{j=1}^{d} \frac{a(x)}{x-r_j}$ as

$$\sum_{j=1}^{d} (-1)^{d-i} \sum_{j_1<\ldots<j_{d-i},\, j_k \neq j} r_{j_1} \cdots r_{j_{d-i}}.$$

Each product $r_{j_1} \cdots r_{j_{d-i}}$ occurs i times, once for each $j = 1, \ldots, d$ not in the set $\{j_1, \ldots, j_{d-1}\}$. Thus, we have that the coefficient is equal to ia_i.

7. We have

$$\frac{a(x)}{x-r_d} \ll \frac{a(x)}{x-r_{d-1}} \ll \ldots \ll \frac{a(x)}{x-r_1} \ll a(x),$$

and these polynomials' leading coefficients all have the same sign. Applying the interlacing pattern from Problem 5,

$$\frac{a(x)}{x-r_d} \ll \frac{a(x)}{x-r_d} + \frac{a(x)}{x-r_{d-1}} \ll \frac{a(x)}{x-r_{d-1}} \ll \frac{a(x)}{x-r_{d-2}}.$$

Iterating, we have $\frac{a(x)}{x-r_d} + \ldots + \frac{a(x)}{x-r_k} \ll \frac{a(x)}{x-r_{k-1}}$, and eventually

$$\frac{a(x)}{x-r_d} + \ldots + \frac{a(x)}{x-r_1} = \sum_{i=1}^{d} ia_i x^{i-1} \ll a(x).$$

(Calculus note: this is saying that the relative maxima and minima of a occur between its roots.)

8. Either person n can be on a team of two or a team of more than two members. In the first case, there are $n - 1$ options for their partner, and $T_{k,\, n-2}$ ways to form k teams from the remaining $n - 2$ people. Thus, we get a contribution of $(n - 1)T_{k,\, n-2}\, x^{k+1}$. In the second case, we first form k teams from $n - 1$ people; then, there are k options for which team to add person n to, so we get a contribution of $kT_{k,\, n-1}\, x^k$. Adding all these contributions together, we obtain the desired recurrence.

Summer Fun!

9. We will do this inductively, while also showing that all roots are nonpositive. For our base case we can check that $T_2 \ll T_3$, and both have a single nonpositive root at 0. Now we assume $T_{i-1} \ll T_i$ for all $i < n$. From Problem 8, we know

$$\sum_{k=1}^{\lfloor (n-1)/2 \rfloor} kT_{k,n-1}x^{k-1} \ll T_{n-1}(x).$$

Since $T_{n-2}(x) \ll T_{n-1}(x)$ by our inductive hypothesis, we have $(n-1)T_{n-2}(x) \ll T_{n-1}(x)$. Thus, we have

$$\sum_{k=1}^{\lfloor (n-1)/2 \rfloor} kT_{k,n-1}x^{k-1} + (n-1)T_{n-2}(x) \ll T_{n-1}(x).$$

Because all of the roots of $\sum_{k=1}^{\lfloor (n-1)/2 \rfloor} kT_{k,n-1}x^{k-1} + (n-1)T_{n-2}(x)$ and $T_{n-1}(x)$ are nonpositive, we have

$$T_n(x) = x\left(\sum_{k=1}^{\lfloor (n-1)/2 \rfloor} kT_{k,n-1}x^{k-1} + (n-1)T_{n-2}(x)\right) \ll T_{n-1}(x),$$

since multiplying by x just introduces an additional root at 0.

10. There are many cool examples. A particularly well-known one is the "Eulerian polynomials."

Given an arrangement of the numbers 1, …, n, the descent number is the number of times a number is bigger than the number immediately to its right. If σ is an arrangement, we denote its descent number by des(σ). For example, des(5764312) = 4 and des(12345) = 0.

We can define polynomials

$$A_n(x) = \sum_{\text{arrangements } \sigma \text{ of } 1,\ldots,n} x^{\text{des}(\sigma)}.$$

It turns out these satisfy a nice recurrence that can be used to show their real-rootedness.

Summer Fun!

Your Mind on the Mind

by Hanna Mularczyk | edited by Jennifer Sidney

1. If I draw x, there are 99 possible cards left, which Anna draws from with equal probability. Of those, $x - 1$ of them are $< x$, and $100 - x$ of them are $> x$. So the probability of drawing one with a value $> x$ is $(100 - x)/99$. As a sanity check, if I draw 1, then it is certain that I have the lower card, and if I draw 100, it's certain I have the higher card.

2. $(100 - 50)/99 = 0.5050\ldots$, while $(100 - 51)/99 = 0.4949\ldots$, so $m = 50$ is the largest such value.

3. To get the probability of winning here, it will be easier to calculate the probability that we lose and then subtract that from 1. If one card is ≤ 50 and the other > 50, we always win, so we only need to consider the cases where either both $x, y \leq 50$, or both $x, y > 50$. Either circumstance has a (1/2)(49/99) chance of occurring, in which case we lose half of the time since we place the cards at random. So the total probability of losing is 2(1/2)(1/2)(49/99) = 49/198, thus the probability of winning is 1 – 49/198 = 149/198 ≈ 0.7525.

4. Anna and I can both start our stopwatches at the beginning of the game, and we each play card x when our watch reads x seconds. This means that the lower card will always be played first. Note that this strategy can extend to arbitrarily many players.

5. We do a similar calculation as in Problem 3, calculating the probability of losing. There are two cases in which we can lose:

- Either both $x, y \leq c$ or both $x, y \geq 101 - c$. In either case, the probability of this occurring is $\frac{c}{100}\frac{c-1}{99}$, in which case we have 1/2 probability of losing.
- We both hesitate, so $c + 1 \leq x, y < 101 - c$. The probability of this occurring is $\frac{100-2c}{100}\frac{100-2c-1}{99}$, in which case we have 1/2 probability of losing.

Together, our probability of losing is

$$\begin{aligned} &\frac{c}{100}\frac{c-1}{99}+\frac{1}{2}\frac{100-2c}{100}\frac{100-2c-1}{99} \\ =\ &\frac{1}{9900}\left(c(c-1)+\frac{1}{2}(100-2c)(100-2c-1)\right), \end{aligned}$$

so the probability of winning is $1 - (3c^2 - 200c + 4950)/9900$.

Summer Fun!

6. If you know calculus, you can solve this by taking the derivative of the winning probability formula from the answer above, with respect to c, and setting it to 0 to find the function's maximum. Using the power rule, the derivative is $-(6c - 200)/9900$. To find the maximum, we must solve the equation $-(6c - 200)/9900 = 0$. We find that $c = 100/3 = 33.\overline{3}$, so our maximum is either when $c = 33$ or $c = 34$.

Plugging $c = 33$ and $c = 44$ into the original winning probability formula gives 8283/9900 and 8282/9900, respectively; so $c = 33$ is our winning solution, giving a winning probability of $8283/9900 = 0.83\overline{6}$ (an improvement from our last strategy!). Note that this makes the "play immediately," "hesitate," and "wait indefinitely" intervals all the same size (give or take, because of rounding).

If you don't know calculus, you could also solve this problem by using the fact that this quadratic formula corresponds to an upside-down parabola with a unique maximum at its vertex; on either side of the vertex, the quadratic decreases as you move away from the vertex.

7. To do this exploration, start by repeating the calculations in Problem 5, but with more cases!

8. For my card, x, to be the lowest, all of the remaining p unplayed cards drawn must have been $> x$. The probability of the first unplayed card being $> x$ is $(100 - x)/(99 - i)$, since there are $100 - x$ cards larger than x and $99 - i$ options for cards larger than the current card on the table (99 – not 100 – because I also have my card). The probability of the next unplayed card being $> x$ is $(99 - x)/(98 - i)$, and so on, giving probability

$$\frac{100-x}{99-i}\frac{99-x}{98-i}\frac{98-x}{97-i}\cdots\frac{100-p-x}{100-p-i}.$$

Another way to calculate this is to notice that the probability is the number of ways to pick p cards from $100 - x$ cards divided by the number of ways to pick p cards from $99 - i$ cards, which is

$$\frac{\binom{100-x}{p}}{\binom{99-i}{p}}$$

and gives the same expansion.

9. Actually, the smallest value of x where the chances are greater than 1/2 that it is the lowest of the unplayed cards is $i + 1$, so let's try to find the largest such value of x. Below is an example of a rough calculation I came up with. First, note that the actual answer will be between i and $101 - p$, and not too close to either extreme.

Summer Fun!

In the expression for the probability,

$$\frac{100-x}{99-i}\frac{99-x}{98-i}\frac{98-x}{97-i}\cdots\frac{100-p-x}{100-p-i},$$

there won't be too much of a difference between the first fraction and the last, assuming that p isn't too big (say, it's in the single digits); and all the other fractions will be in between these extremes. So, we'll approximate the probability by replacing each fraction with the fraction

$$\frac{100-p/2-x}{99-p/2-i},$$

which gives us the approximation $\left(\frac{100-p/2-x}{99-p/2-i}\right)^p$ for the probability.

Setting this equal to 1/2 and solving for x yields $x \approx 100 - p/2 - (1/2)^{1/p}(99 - p/2 - i)$. (For example, if $p = 4$ and $i = 18$, then $x = 31$ and our approximation gives 31.569…)

10. Correction: in this question, the second instance of "cutoff value" should say "probability." This singular round is won if exactly one player has a card before the cutoff value (otherwise, the round can still be won if there is a tie and it happens to be played in the correct order, but we will ignore these smaller terms in this calculation). Set $c = 100 - p/2 - (1/2)^{1/p}(99 - p/2 - i)$. The probability that one of the drawn cards with value $> i$ is $\leq c$ is $(c - i)/(100 - i)$. Then the remaining cards with value $> i$ must have value greater than c, which has probability

$$\frac{100-c}{100-i}\frac{99-c}{100-i}\frac{98-c}{100-i}\cdots\frac{100-p-c}{100-i}.$$

Multiplying these all together gives the probability, though in this form it is not very illuminating.

11. My random number generator picked 5, 40, 78, and 93. At the beginning of the game $i = 0$ and, in the eyes of each player, $p = 3$. Plugging this into the solution to Problem 9, the cutoff value is about 21. So the player with card 5 will play, and none of the other players will. Now, $i = 5$ and $p = 2$, so plugging this in again gives a new cutoff of about 33. Since 40 > 33, the player with card 40 won't play right away; eventually, someone will give in at random, so now the chances of winning are diminished to 1/3. In the case that the next card is put down correctly, we have cards 78 and 93 remaining, with $i = 40$ and $p = 1$, and the new cutoff is about 70. Since 78 > 70, the player with 78 won't play right away, and again it's a tossup which of the two remaining players will play first. So in the end there is only a $1/3 \times 1/2 = 1/6$ chance of winning. Upgrading to a c-hesitation strategy in this setting could help fix this. That said, this is still an improvement over playing the cards in a random order, in which case we only win with probability $1/(4!) = 1/24$.

Summer Fun!

Calendar

Session 37: (all dates in 2025)

September	11	Start of the thirty-seventh session!
	19	
	25	
October	2	
	9	
	16	
	23	
	30	
November	6	
	13	
	20	
	27	Thanksgiving - No meet
December	4	

Girls' Angle has run over 150 Math Collaborations. Math Collaborations are fun, fully collaborative, math events that can be adapted to a variety of group sizes and skill levels. We now have versions where all can participate remotely. We have now run four such "all-virtual" Math Collaboration. If interested, contact us at girlsangle@gmail.com. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

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Girls’ Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant’s Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ____________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don’t like math, what don’t you like? If you love math, what do you love? What would you like to get out of a Girls’ Angle Membership?

____-

The $50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- ☐ Enclosed is a check for $50 for a 1-year Girls’ Angle Membership.
- ☐ I am making a tax-free donation.

Please make check payable to: **Girls’ Angle**. Mail to: Girls’ Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

Girls' Angle

A Math Club for Girls

Girls' Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is \$20/meet for members and \$30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle is based in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, founder and director of the Exploratory
Yaim Cooper, Institute for Advanced Study
Julia Elisenda Grigsby, professor of mathematics, Boston College
Kay Kirkpatrick, associate professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, assistant dean and director teaching & learning, Stanford University
Lauren McGough, postdoctoral fellow, University of Chicago
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, associate professor, University of Utah School of Medicine
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Liz Simon, graduate student, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, associate professor, University of Washington
Karen Willcox, Director, Oden Institute for Computational Engineering and Sciences, UT Austin
Lauren Williams, professor of mathematics, Harvard University

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high-level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: _______________________

(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is $216 for one session (12 meets)	☐ I will pay on a per meet basis at $30/meet.
☐ I will pay on a per meet basis at $20/meet.	☐ I'm including $50 to become a member, and I have selected an item from the left.

☐ I am making a tax-free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: ________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____________________________________