Girls' Angle Bulletin

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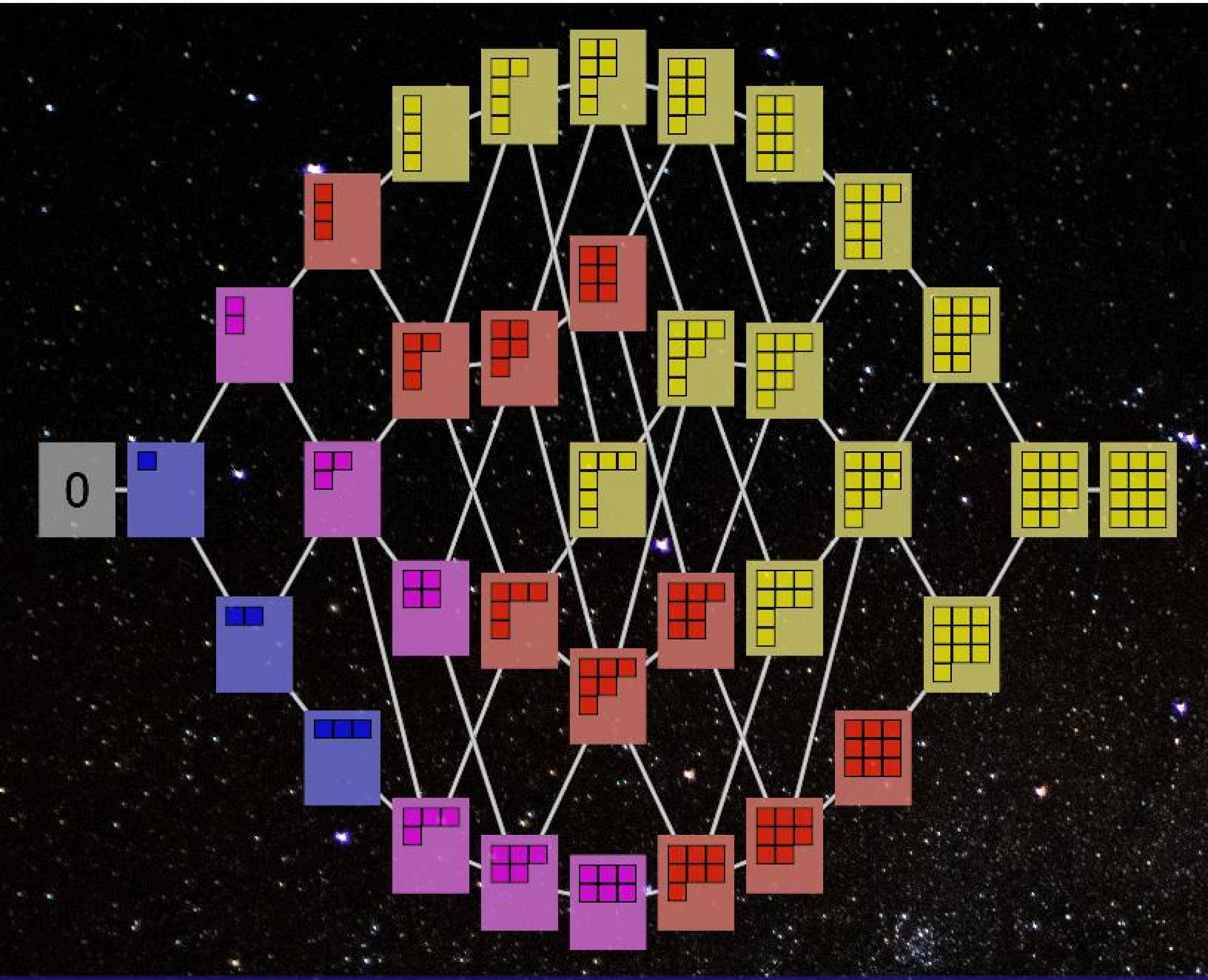
To Foster and Nurture Girls' Interest in Mathematics



From the Founder

Much content in this Bulletin is focused on doing math. Why? Because our members have generally responded well to approaching math as a creative subject. The club is a mathematical greenhouse seeded by member's thoughts and tended to by them as well. - Ken Fan, President and Founder

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Girls' Angle Bulletin

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Girls' Angle welcomes submissions that pertain to mathematics.

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Editors: Amanda Galtman
Jennifer Sidney
Executive Editor: C. Kenneth Fan

Girls' Angle: A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: $L(4, 3)$ by Robert Donley. Background starry sky image courtesy of NASA. See page 8.

An Interview with Karen Lange, Part 2

This is the second part of our four-part interview with Prof. Karen Lange of Wellesley College.

Ken: I'm fascinated by your answers, and I'm very eager to ask you this: What is the goal of mathematics, and what is the goal of math education, and how are they related?

Karen: Well, I want to know your answers to these questions, too. The goal of mathematics? I would say for me there are many very primary goals of mathematics. I'm going to borrow some understanding from the mathematician Eugenia Cheng's work.

One thing that she might say is, "Mathematics is about understanding truth as determined by logic." That truth is determined by logical rules of inference from basic axioms, which are inspired by patterns we see in the real world. But that starts to feel very technical and epistemological.

But a crucial part in that is the patterns part: What are the patterns that we see about abstracted objects in the world, whatever that means, right? And then, what can we say about those abstracted ideals using logic? That, to me, is the goal of mathematics.

And so, at a very meta level, mathematics is understanding truth. Any discipline is trying to understand truth, but I like Eugenia Cheng's description about using the tools of logic and understanding abstracted ideas, things that we see from the concrete world, and then applying logic to see what's true via logic.

And sometimes that can lead you to applied math or understanding theoretical physics, but sometimes it leads you just to its own beauty, because it's true. Maybe your axioms — your starting places — have gotten so generalized away from real contexts that now you're just seeing what grows out of those properties. And as we know, sometimes that ties back to the real world in ways we didn't expect up front. So, I think that's the goal of mathematics for me.

I think that's a fundamental part of mathematics: the building and finding truth via logic in community with others.

I would also say that working toward understanding truth at this meta level is a fundamentally human activity. Humans want to know what's true. For me, there's equal importance in doing mathematics and the community together doing mathematics. You could have a view of mathematics of, "Oh, just understanding truth is all mathematics is." But for me, it's also understanding and building a body of knowledge together with others. That human aspect to it is very fundamental to me.

After learning something myself, what did I do with my discovery? I went to my math teacher, and said, "There's a formula here. There's something here!" And I think that's a fundamental part of mathematics: the building and finding truth via logic in community with others.

So, what's mathematics education? I think it's about giving people who are earlier on their mathematical journey — I almost don't like the word "students," because we're all constantly students, especially in mathematics — the tools and experiences, an apprenticeship into this community.

That wording bothers me a little bit, because this community is for everyone. Everyone is part of it, but sometimes people who are farther along in their mathematical journeys, we don't always help give people earlier on their journeys all the tools that they need, and sometimes we do. Mathematics education requires thinking about how we can give them people tools and experiences to help them develop their own mathematical agency.

Ken: That's beautiful. I love that. It's inspiring. Actually, I got a little chill on my spine there, when you talked about community and that it's a human endeavor.

Karen: Exactly, and it's such an honor for us in what we do. It's such an honor to be able to help people on their mathematical journeys. Yes, I learned a lot of math in grad school, and that was lovely. But where did I really find my power? It was really earlier in my journey. Those are such special moments, and to get to be a part of that with other people earlier in their journeys is a gift.

Mathematics education requires thinking about how we can give people tools and experiences to help them develop their own mathematical agency.

Ken: Can you recap your own journey to being a professional mathematician?

Karen: Sure. So, I was raised just outside Milwaukee, Wisconsin. In grade school, I went to a Catholic school, and I liked math, but I liked everything. I wasn't "math kid."

Opportunities now are a little different from when I was growing up. Now, I think math is more of a thing. We think kids need to really learn more math. But then, in high school, I had this wonderful class in ninth grade with Mr. Gustafson, and that was my moment where he had given me these experiences and these tools to realize, "Oh, I can do all this stuff." So, that was when I bought into, "Hey, math is really cool." But again, I had lots of interests. I loved to do theater. I loved to play tennis. I loved a lot of other stuff.

But my high school had some very informal math activities — like a quick 20-minute math competition-y kind of thing after school. And I don't know why I started — I think because it was fun. It was a fun thing to do, and in high school there were opportunities to take the AMC.[1] But it wasn't an emphasis, it was just, "Oh, these are fun problems." So, I went and did the problems.

I was fortunate that they went well for me. That made me feel good and gave me more confidence, but I can also totally see how if I didn't do so well in them, I would have been less likely to engage. So, I think about that a lot with my students: math competitions can be fun for those people who like that kind of math, but there are all sorts of other math that's not competition. But that was the start for me. Then, through competitions, my teacher started to say, "You know, there are more competitions," and I qualified for some next-level things.

I admit, that was very eye-opening. I didn't realize there was a world of math competitions. I went to a state competition and met other kids who practiced for these things. It just wasn't on my radar earlier. And so, I did a little bit of that, but I didn't do a lot. And actually, to be honest, my favorite competitions were like what you sometimes run, or things like, "Here's a problem of the month. Send in your answer." There would be no time bounds,

[1] The AMC is short for American Mathematics Competition. There are actually two, the AMC 10 and the AMC 12, both organized by the Mathematical Association of America.

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Thank you and best wishes,
Ken Fan
President and Founder
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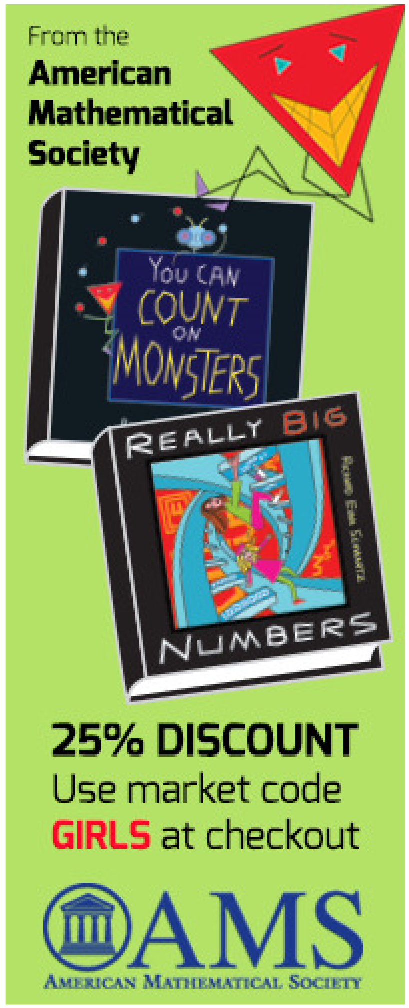

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A Partial Order for Partitions

by Robert Donley[1]
edited by Amanda Galtman

In the previous installment, we used Young diagrams as a visual aid to study partitions and developed methods for counting partitions. In this installment, we apply the approach for compositions from "*n*-Cubes and Up Operators" (see Volume 16, Number 2 of the Girls' Angle Bulletin) to partitions; in particular, partial orderings, Hasse diagrams, and up operators help us organize partition data and give context to path counting problems.

We keep the definitions and notation from previous installments; in particular, recall that partitions are denoted by non-increasing strings of numbers (called the parts of the partition). It will sometimes be convenient to denote partitions with trailing zeros in definitions and figures.

Definition: Let P be the set of all partitions. Let $\mu = \mu_1 \dots \mu_k$ and $\lambda = \lambda_1 \dots \lambda_k$ be partitions with k parts. We define a partial ordering on P by the rule that $\mu \leq \lambda$ if $\mu_i \leq \lambda_i$ for all i. When partitions are extended by zeros, any two partitions can be compared for some k. Alternatively, $\mu \leq \lambda$ if the corresponding Young diagram for μ fits inside that for λ.

The partially ordered set $(P, \leq)$ is called the **Young lattice**. In fact, our interest lies only in finite subsets of the Young lattice.

Exercise: Draw as much of the Young lattice as you can by hand, using either the digit string notation or Young diagrams.

Definition: Suppose $m, n \geq 1$. Denote by $L(m, n)$ the subset of all partitions λ with at most m parts such that each part $\lambda_i \leq n$. Alternatively, the corresponding Young diagrams fit inside a rectangle with width n and height m.

To obtain a symmetric Hasse diagram for each $L(m, n)$, we include the empty partition, which we denote by 0. The **rank** of a partition is the sum of its parts. The empty partition has rank 0, and partitions with the same rank are at the same level of the Hasse diagram of $L(m, n)$.

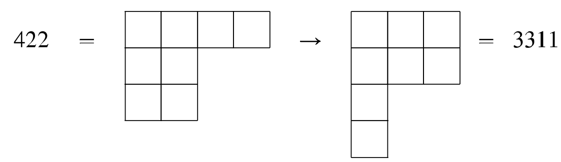
Definition: Recall that we say λ **covers** μ if $\mu \leq \lambda$ and there are no other elements between μ and λ. The partition λ has the same parts as μ except in one entry. The Young diagram for λ is obtained by adding one square to the Young diagram for μ.

Example: $L(m, 1)$ and $L(1, n)$ are simply chains. Oriented horizontally, the Hasse diagrams for $L(4, 1)$ and $L(1, 4)$ are

0 — 1 — 11 — 111 — 1111 and 0 — 1 — 2 — 3 — 4.

In the last installment, we noted the operation of conjugation on partitions, which interchanges the rows in the Young diagram with its columns. For example, the conjugate of 422 is 3311, as seen in the following figure:

[1] This content is supported in part by a grant from MathWorks.

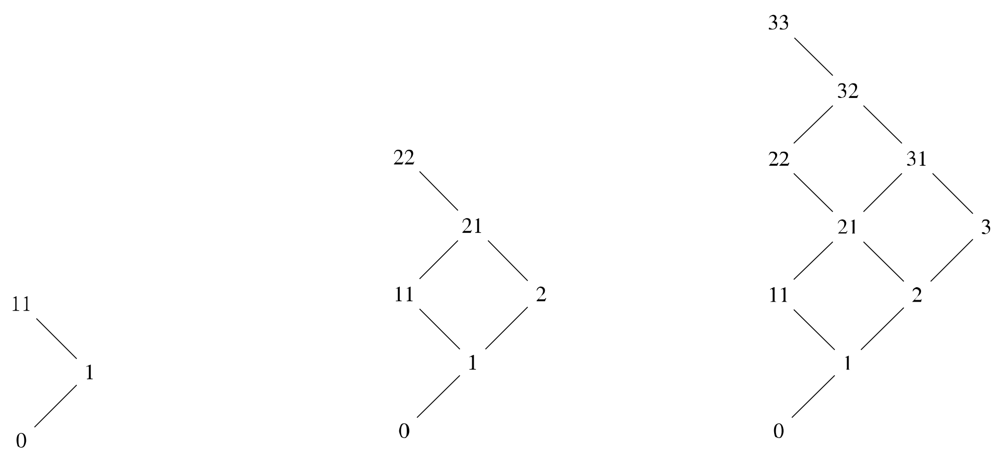


Definition: We denote the conjugate of λ by λ^*.

Exercise: Prove that if $\mu \leq \lambda$ then $\mu^* \leq \lambda^*$. Note that conjugation preserves rank.

Note that the conjugate of a rectangular Young diagram is another rectangular Young diagram, so conjugation interchanges the elements of $L(m, n)$ and $L(n, m)$ while preserving the partial orderings. In particular, conjugation carries $L(m, m)$ back into itself.

Example: Consider the Hasse diagrams for $L(2, 1)$, $L(2, 2)$, and $L(2, 3)$:



We encountered $L(2, n)$ before in "Examples of Posets" (see Volume 16, Number 1 of the Girls' Angle Bulletin); these Hasse diagrams are associated with the Catalan numbers. To obtain $L(2, n + 1)$ from $L(2, n)$, we append a chain with $n + 2$ nodes along the top of the diagram. This construction reflects the recurrence relation on generating functions from the previous installment:

$$F_{m,n}(t) = F_{m,n-1}(t) + t^n F_{m-1,n}(t).$$

Recall that $F_{m,n}(t)$ is the generating function that counts partitions in $L(m, n)$; in the language of posets, the coefficient of t^k counts the number of partitions of rank k in $L(m, n)$. When $m = 2$,

$$F_{2,n+1}(t) = F_{2,n}(t) + t^{n+1} F_{1,n+1}(t).$$

The second term indicates that the bottom of the chain is placed at level $n + 1$.

Exercise: Redraw the Hasse diagrams above by replacing each node with the corresponding Young diagram. Then draw the diagram again by replacing each Young diagram with its conjugate. Finally, replace these Young diagrams with the corresponding strings of digits. How does conjugation change the property of having at most two parts?

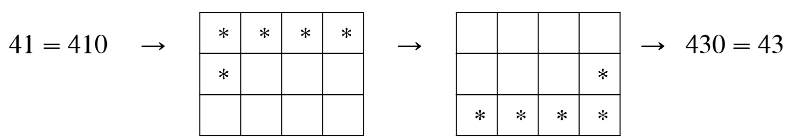
Exercise: Draw the Hasse diagrams for $L(2, 5)$ and $L(2, 6)$. For $n \leq 6$, verify the counts for each rank in $L(2, n)$ with the coefficients of $F_{2,n}(t)$ from the previous installment.

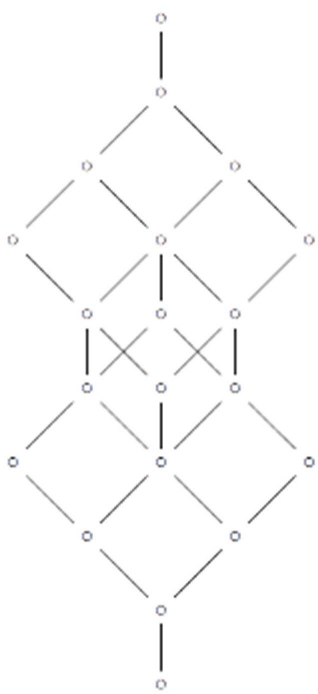
Exercise: Describe the first and last three levels of any $L(m, n)$ with $m, n \geq 2$. Verify using Young diagrams.

In addition to conjugation, we define another operation on partitions. Unlike conjugation, this other operation depends on the choice of $L(m, n)$.

Definition: For the partition λ in $L(m, n)$, the **complement** of λ is the partition obtained by replacing each part λ_i with $n - \lambda_i$ and reversing the order of this string of numbers. Here we include zero as a part if needed.

For instance, the complement of (4, 1, 0) in $L(3, 4)$ is (4 – 0, 4 – 1, 4 – 4) = (4, 3, 0). To express complementation with Young diagrams, we mark the squares of the partition in the rectangle, rotate the rectangle by 180°, and remove the marked squares. With Young diagrams:





Exercise: In $L(3, 5)$, compute the complements of 543, 411, and 33 numerically. Then verify using the procedure for the corresponding Young diagrams.

Exercise: In $L(m, m)$, prove that the complement of the conjugate is the conjugate of the complement. How can we extend this result to general $L(m, n)$?

Exercise: What is the complement of the complement? What conditions are necessary for a partition to be equal to its complement? To the conjugate of its complement?

Exercise: Fill in the template for $L(3, 3)$ shown at left with both the corresponding partitions and Young diagrams. Identify nodes that are fixed under conjugation. Match nodes that are paired under conjugation. Finally, match nodes that are paired under complementation.

Exercise: The recurrence relation suggests that this Hasse diagram splits into two copies of $L(2, 3)$. Express this splitting by coloring the nodes of the diagram based on whether 3 is a part in the partition. Describe the new splitting if we replace each node with its conjugate.

Exercise: Construct the Hasse diagram for $L(3, 4)$ by appending the diagram for $L(2, 4)$ to $L(3, 3)$. Represent $L(2, 4)$ within the $L(3, 4)$ diagram by those partitions with at least one part equal to 4; that is, construct $L(2, 4)$ as usual and precede each entry with a 4. Match complements in the diagram. What happens if we try to match conjugates?

Exercise: Verify the coefficients of $F_{3,3}(t)$ and $F_{3,4}(t)$ by counting the nodes in each level of the Hasse diagrams. In all of the examples above, verify that the number of elements in $L(m, n)$ is $\binom{m+n}{m}$.

The rank numbers of $L(m, n)$ are the beginning of the theory of q-binomial coefficients, defined below, also known as Gaussian polynomials. By tradition, we use the variable q instead of t.

Definition: For $N \geq 1$, we define the q-**factorial** as

$$[N]_q! \equiv (1)(1+q)(1+q+q^2)\cdots(1+q+\ldots+q^{N-1}).$$

As with the usual factorial, we define $[0]_q! = 1$. We define the q-**binomial coefficient**

$$\begin{bmatrix} m+n \\ m \end{bmatrix}_q = \frac{[m+n]_q!}{[m]_q![n]_q!}.$$

Exercise: What happens to $[N]_q!$ and $\begin{bmatrix} m+n \\ m \end{bmatrix}_q$ when $q = 1$? Give an interpretation of $[N]_q!$ as a generating function in the variable q.

Exercise: Prove that $F_{m,n}(q) = \begin{bmatrix} m+n \\ m \end{bmatrix}_q$ in the specific cases above and, in particular, $F_{2,n}(q)$ as expressed in the previous installment. Prove this equality in general by showing that $\begin{bmatrix} m+n \\ m \end{bmatrix}_q$ satisfies the same recurrence relation as $F_{m,n}(q)$.

Exercise: Can you prove the following equality directly?

$$[m+n]_q! = F_{m,n}(q)[m]_q![n]_q!$$

Properties of $F_{m,n}(q)$ show that the q-binomial coefficients are in fact polynomials in q and that their coefficients are symmetric. For further results on q-binomial coefficients and for partitions in general, we highly recommend the book *Integer Partitions* by George E. Andrews and Kimmo Eriksson.

We continue to pursue analogies to compositions with path counting and the up operators for $L(m, n)$. For lattice path counting, we count (**saturated) chains** from 0 to λ. These are sequences from 0 to λ such that each element is covered by the next element in the sequence. For example, here is a chain from 0 to 321:

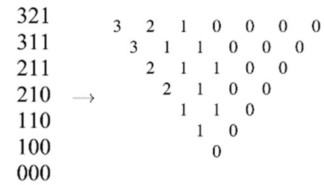
$$0 — 1 — 11 — 21 — 211 — 311 — 321.$$

Definition: Let f^{λ} denote the number of chains from 0 to the partition λ.

Exercise: Find f^{λ} for each λ in $L(2, 2)$ and $L(2, 3)$ by listing all chains.

Exercise: Trace the chain to 321 in your picture for $L(3, 3)$. List as many chains as you can from 0 to 321. Can you find f^{λ} for each element in $L(3, 3)$ without listing all chains?

Working with chains directly can be cumbersome. As a first attempt at organization, we simply record the sequence as a list with trailing zeros. Further, we see that any such sequence can be fitted to an equilateral triangle by justifying the data to the leftmost diagonal.

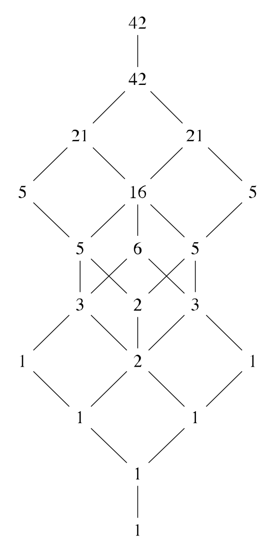


We note some properties of such a triangle: each row is non-increasing, as is each diagonal to the lower right. Also, as rows progress downwards, a single entry in each row is reduced by 1 such that the non-increasing property is preserved. Thus the sum of the row entries yields the sequence k, …, 2, 1, 0. Finally, each entry is between the two entries in the row above. These triangles are **Gelfand-Tsetlin patterns** from the previous installment, although now we include zeros as entries.

Exercise: Suppose λ is a partition of k. Prove that any chain from 0 to λ can be fitted to an equilateral triangle as above with side length $k + 1$. Can you relate the number of parts of λ to zeros in the triangle? Which zeros in the pattern indicate that the number of parts in the sequence has changed?

Exercise: List all such Gelfand-Tsetlin patterns for all chains from 0 in $L(2, 2)$ and $L(2, 3)$.

Exercise: Prove all properties in the paragraph before the preceding exercises. What pattern has the most zeros? The fewest zeros? List the corresponding chains and draw these chains with the corresponding Young diagrams.



Similar to an n-cube, $L(m, n)$ is a **graded** poset; that is, any chain from 0 to λ has the same number of elements. Then every nonzero partition covers an element on the level below it and no partition covers an element two or more levels below it. This property allows for a uniform approach to chain counting, as seen in "*n*-Cubes and Up Operators."

If the diagram for $L(m, n)$ is available, then we use the corresponding version of Pascal's identity: to count the number of chains leading to λ, we sum f^{μ} over all μ covered by λ.

Exercise: Verify Pascal's identity for each element of $L(3, 3)$ in the diagram on the right. Find f^{λ} for each element of $L(3, 4)$ using Pascal's identity.

When the Hasse diagram is not available, we refer to linear algebra and the up operator.

Example: We apply the up operator to recover the first few levels in $L(3, 3)$. Recall that the f^{λ} values for level k are calculated with $U^k[000] = U \cdots U[000]$ (k times). Now U operates by adding one to parts only where that addition maintains the non-increasing property. By applying linearity rules as seen in (installment), we obtain

$$\begin{aligned} U[000] &= [100], \\ U^2[000] &= U[100] = [200] + [110], \\ U^3[000] &= U(U^2[000]) = U[200] + U[110] \\ &= ([300] + [210]) + ([210] + [111]) = [300] + 2[210] + [111], \end{aligned}$$

and so on.

Exercise: Complete the calculations of $U^k[000]$ for $L(3, 3)$. Then use this method to calculate each f^λ for $L(3, 4)$ and $L(4, 4)$. Use these values to construct the Hasse diagram for $L(4, 4)$, and then verify the q-binomial coefficients and the split into two $L(3, 4)$ diagrams.

Exercise: For $L(m, n)$, prove that $f^{n \dots n}$ is the sole coefficient in $U^{mn}0$.

Exercise: Find a formula for f^λ if λ has two parts. How about three parts?

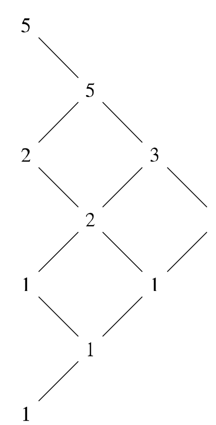
If we record each f^λ as a node of the Hasse diagram for $L(m, n)$, we obtain a version of

Chu-Vandermonde convolution in $L(m, n)$: Choose k satisfying $0 \le k \le mn$. Then $f^{n \dots n}$ equals the sum of all products $f^\lambda f^\mu$ over all partitions λ of k, where μ is the complement of λ.

To implement the convolution, k indicates a choice of level in the Hasse diagram of $L(m, n)$. As seen in "Examples of Posets," each chain from 0 to $n \dots n$ passes through one and only one node in level k, so the number of all such chains splits into a sum over the nodes at level k. On the other hand, the number of such chains that pass through λ is the product of the number of chains from 0 to λ with the number of chains from λ to $n \dots n$. By the symmetry of the Hasse diagram, the latter count is the number of chains from 0 to the complement of λ.

Example: For $L(2, 3)$, complements are paired vertically on levels symmetric about the central horizontal axis. Then we have

$$5 = 5 \times 1 = 2 \times 1 + 3 \times 1 = 2^2 + 1^2.$$



Example: For $L(3, 3)$, we can arrange that complements again pair vertically, so that the diagram for $L(3, 3)$ gives the equalities

$$\begin{aligned} 42 &= 42 \times 1 = 21 \times 1 + 21 \times 1 \\ &= 5 \times 1 + 16 \times 2 + 5 \times 1 \\ &= 5 \times 3 + 6 \times 2 + 5 \times 3. \end{aligned}$$

Exercise: Verify the convolution property for $L(3, 4)$. A key step is to identify the complements for each level.

At this point, it is unclear whether we should expect a reasonable general formula for f^λ. On one hand, the corresponding numbers for compositions are readily computed. But we have also seen that partitions are far more difficult to work with than compositions. In fact, such a formula exists and, in the next installment, it will be Young diagrams that provide the closed formula for f^λ in a most unexpected and satisfying way.

Follow Your Nose, Part 2

by Ken Fan | edited by Jennifer Sidney

When you follow your nose in math, not only might you figure out things without consulting any books, but you might also come up with something not commonly known … or not known at all!

In the first part, we followed a group of 8th graders who took it upon themselves to figure out a way to generate all of the Pythagorean triples. After they succeeded, I showed how to extend their results by identifying when their method produces primitive Pythagorean triples. In the process, we found that in any primitive Pythagorean triple a, b, c with $a^2 + b^2 = c^2$, the differences $c - a$ and $c - b$ give us an odd perfect square and twice a perfect square, in some order.

However, their method raises a natural question that we left unresolved. Their method produces each primitive Pythagorean triple twice, in the sense that both the primitive Pythagorean triple a, b, c and b, a, c are produced from different parameters. Can we figure out conditions on the parameters so that only one of these closely-related triples is produced?

Before proceeding, let's review the method:

> To produce a primitive Pythagorean triple, pick either an odd perfect square or twice a perfect square and call it x. Pick any positive integer m that is relatively prime to x.
>
> Let r_0 be the smallest positive integer whose square is divisible by $2x$.
>
> Then
>
> $$a = mr_0 + x \qquad b = mr_0(mr_0 + x)/(2x) \qquad c = b + x$$
>
> is a primitive Pythagorean triple with $a^2 + b^2 = c^2$.

For example, let's take x to be 8 and m to be 5. Then $r_0 = 4$ and $(a, b, c) = (28, 45, 53)$.

But look what happens if we take x to be 25 and m to be 2. In that case, we find $r_0 = 10$ and $(a, b, c) = (45, 28, 53)$, which is essentially the same triple we just found.

What can we do to ensure that of the two triples a, b, c and b, a, c, we only produce one?

A natural way to proceed would be to insist that $a < b < c$, and then try to determine which parameters x and m result in triples that satisfy this inequality.

However, what's special about these 8th graders' method is that one of the parameters, x, corresponds to the difference $c - b$; and we saw last time that their method almost effortlessly leads to the observation that in any primitive Pythagorean triple, the differences between the hypotenuse and each leg length yield an odd perfect square and a number that is twice a perfect square. Switching the leg lengths simply flips which of the numbers $c - a$ and $c - b$ is the odd perfect square and which is twice a perfect square.

So all we have to do to ensure that we do not get the same Pythagorean triple with its leg lengths flipped is to insist that x be either an odd perfect square or twice a perfect square! Since "odd perfect square" is a little easier to say than "twice a perfect square," let's go ahead and insist that x be an odd perfect square (it doesn't really matter which we pick, so long as we stay consistent).

To summarize, rather than take the half of the primitive Pythagorean triples a, b, c where $a < b$, we take the half of the primitive Pythagorean triples where $c - b$ is an odd perfect square.

Since we know x is an odd perfect square, let's write $x = n^2$, where n is an odd number, and rewrite the formulas for our Pythagorean triple in terms of n and m instead of x and m. (Note that m is relatively prime to x if and only if m is relatively prime to n.) We find that $r_0 = 2n$.

Then $a = mr_0 + x = 2mn + n^2$.

And $b = mr_0(mr_0 + 2x)/(2x) = 2mn(2mn + 2n^2)/(2n^2) = 2m(m + n) = 2mn + 2m^2$.

And $c = b + x = 2mn + 2m^2 + n^2$.

Notice that $c - b = n^2$, as expected, and $c - a = 2m^2$. That is, because we picked x to be an odd perfect square, by design, we know that $c - b$ will be that odd perfect square; but now we also know precisely which perfect square $c - a$ is twice of. Also, the formulas completely eliminate r_0 from the picture! That is, all we have to do to generate primitive Pythagorean triples is pick a positive odd number n and a positive number m relatively prime to n. We then compute $2nm$, then form the three right triangle side lengths by adding to $2nm$ either n^2, $2m^2$, or the sum of both, $2m^2 + n^2$, and we have a primitive Pythagorean triple!

To illustrate, take $n = 3$ and $m = 2$. Then $2nm = 12$. Our triple is

$$12 + 3^2,\ 12 + 2(2^2),\ 12 + 2(2^2) + 3^2$$

or 21, 20, 29.

Or take $n = 11$ and $m = 7$. Then $2nm = 154$. Our triple is

$$154 + 11^2,\ 154 + 2(7^2),\ 154 + 2(7^2) + 11^2$$

or 275, 252, 373.

Isn't this a nifty way to generate the primitive Pythagorean triples? It differs from Euclid's method, which is the standard one that is explained in books, and it doesn't seem to be in the Wikipedia entry, "Formulas for generating Pythagorean triples" (as of December 22, 2023).

The formulas also show us how to decide, for any primitive Pythagorean triple a, b, c, which of $c - a$ or $c - b$ is the odd perfect square, for c must be odd and only one of a or b will be even. The odd perfect square will be the difference between the hypotenuse and the even leg length.

For a fun algebra exercise, verify the algebraic identity

$$(2mn + n^2)^2 + (2mn + 2m^2)^2 = (2mn + 2m^2 + n^2)^2.$$

a < b < c

Our new way of generating primitive Pythagorean triples makes it straightforward to figure out whether $a < b$ or $b < a$. We just check to see if $n^2 < 2m^2$. If it is, then $a < b$; otherwise $b < a$.

Flips

In order to make this section clearer, let's introduce functional notation for the 8th graders' method described in the yellow box on the first page of this article. That is, given x and m as explained there, let $P(x, m) = (a, b, c)$ where a, b, c is the corresponding Pythagorean triple.

Let's figure out how the parameters must change when we flip a, b, c to b, a, c. That is, we just found the Pythagorean triple associated to the parameters $x = n^2$ and m, where n is a positive odd integer and m is a positive number relatively prime to n, namely $P(x, m)$. What parameters x' and m' correspond to swapping the leg lengths? (That is, so that $P(x, m)$ and $P(x', m')$ differ only in that they swap each other's leg lengths.)

Since x is the difference between c and b, after we flip, the difference between the "new" c and b will be the difference between the "old" c and a; so we must take x' to be $2m^2$, in which case r_0 will be $2m$, and we must pick m' to be such that $2m'(2m) + 2m^2 = 2mn + 2m^2$. Solving for m', we find $m' = n$.

Thus, $P(n^2, m)$ and $P(2m^2, n)$ are the flips of each other.

Here's a summary of this flip (we introduce the letters s and t in order to avoid confusion with the already used m):

<table>
<tr><td colspan="3">Pick any positive odd number s and any positive number t relatively prime to s.</td></tr>
<tr><td>Parameters:

Let $x = s^2$.
Let $m = t$.

Primitive Pythagorean triple:

$a = 2st + s^2$
$b = 2st + 2t^2$
$c = 2st + 2t^2 + s^2$</td><td>← Flip! →</td><td>Parameters:

Let $x = 2t^2$.
Let $m = s$

Primitive Pythagorean triple:

$a = 2st + 2t^2$
$b = 2st + s^2$
$c = 2st + s^2 + 2t^2$</td></tr>
</table>

An Iterative Game

Notice that for the parameters n^2 and m (where n is odd and m is positive and relatively prime to n), we get a primitive Pythagorean triple a, b, c where a is odd and b is relatively prime to a. That means we can apply the method again to the parameters a^2 and b instead of n^2 and m, and we can repeat this process over and over again to our hearts' content!

In other words, our new way of generating primitive Pythagorean triples iterates to give us sequences of primitive Pythagorean triples! Let's see what sequence of triples we get if we start with $n = 1$ and $m = 1$:

n*, *m	***a*, *b*, *c***
1, 1	3, 4, 5
3, 4	33, 56, 65
33, 56	4785, 9968, 11057
4785, 9968	118289985, 294115808, 317012033
118289985, 294115808	83574429584465985, 242590126064151488, 256582646615451713

The lengths grow quickly!

What can you say about these sequences of primitive Pythagorean triples?

Keep in mind that this is all the result of a group of 8th graders who gave themselves a chance to figure something out without looking it up!

Follow *Your* Nose

Did anything in the above development arouse your curiosity? If so, get out some scratch paper and follow up on that curiosity immediately! See what you can make of it.

If not, here are some things to think about:

1. Fix a positive odd number n and consider the sequence of primitive Pythagorean triples given by $P(n^2, m)$, where m takes on all the positive integers relatively prime to n in numerical order. What can you say about this sequence of right triangles? What do their acute angles converge to as m tends to infinity?

2. For a fixed positive odd number n, for which m, relatively prime to n, does $P(n^2, m)$ yield a right triangle that is closest to being isosceles?

3. Can you devise a method for producing all solutions to the equation

$$a^2 + b^2 + c^2 = d^2$$

in positive integers a, b, c, and d? Can you figure out how to produce primitive solutions to this equation (i.e., solutions a, b, c, and d such that there does not exist an integer greater than 1 which divides evenly into all four values)? What patterns can you find in the primitive solutions?

Romping Through the Rationals, Part 6

by Ken Fan | edited by Jennifer Sidney

Emily and Jasmine are studying sequences a_n of nonnegative integers that have the property that consecutive terms are relatively prime and every nonnegative rational number is equal to a_n/a_{n+1} for a unique n. They have dubbed these sequences "rational rompers."

Last time, they showed that any rational romper can be transformed into any other rational romper via a sequence of operations that they called a "splice." They were hoping that a similar statement was true of finite rational rompers, but they found a counter-example instead. Can they salvage their idea?

Recall that a splice modifies a rational romper a_n in the following way: Suppose x, y are consecutive terms in the sequence a_n and suppose there is a subsequence disjoint from the consecutive terms x, y, but which also begins with x and ends with y. Then one can remove the terms between this x and y and reinsert them between the consecutive x and y.

Jasmine: In my counterexample, we can drop the last 4 from both sequences to get the smaller counterexample

$$0, 1, 2, 1, 3, 1$$

and

$$0, 1, 3, 1, 2, 1.$$

Emily: You're right. Both sequences represent the same rational numbers, but there's no way to perform even a single splice to either sequence. To change one into the other, we'd want to swap the 2 and the 3. Swapping the 2 and 3 is kind of like our splice, only instead of moving a sequence situated between an x and y to sit between a consecutive occurrence of x and y, we are swapping the contents of two strings that have the same beginning and ending. Perhaps if we extend our definition of splice to include this kind of swapping operation, we can use them to transform any two finite rational rompers that represent the same rational numbers into each other?

Jasmine: Maybe—let's see! So you want to define a splice as follows: Suppose inside the sequence there are two nonoverlapping subsequences of consecutive numbers that begin and end the same way. We then swap the contents like this:

$$\ldots, \boldsymbol{x}, a_1, a_2, a_3, \ldots, a_n, \boldsymbol{y}, \ldots, \boldsymbol{x}, b_1, b_2, b_3, \ldots, b_m, \boldsymbol{y}, \ldots$$
$$\downarrow$$
$$\ldots, \boldsymbol{x}, b_1, b_2, b_3, \ldots, b_m, \boldsymbol{y}, \ldots, \boldsymbol{x}, a_1, a_2, a_3, \ldots, a_n, \boldsymbol{y}, \ldots$$

Emily: Yes, although we might as well also include the possibility that $x = y$ and the end of the first subsequence is also the beginning of the second:

$$\ldots, \boldsymbol{x}, a_1, a_2, a_3, \ldots, a_n, \boldsymbol{x}, b_1, b_2, b_3, \ldots, b_m, \boldsymbol{x}, \ldots$$
$$\downarrow$$
$$\ldots, \boldsymbol{x}, b_1, b_2, b_3, \ldots, b_m, \boldsymbol{x}, a_1, a_2, a_3, \ldots, a_n, \boldsymbol{x}, \ldots$$

That's the kind of splice we'd need to transform one of your counterexample sequences into the other. In your counterexample, we would take $x = 1$, n and m would both be 1, and $a_1 = 2$ and $b_1 = 3$.

Jasmine: This is an extension of what we were doing because our original swap corresponds to the case where one of n or m is equal to 0.

Emily: You mean "splice…"

Jasmine: You're right, our original *splice*, although this more general operation feels a bit like a *swap* to me.

Emily: It does. So, let's call it a "swap." The question is, if we have two finite rational rompers that represent the same rational numbers and end with the same number, can one be transformed into the other via a finite sequence of swaps?

Jasmine: Let's attempt to prove it along the lines of what we've been trying and see how far we can get.

Emily: Okay. So suppose we have two finite rational rompers that represent the same rational numbers and end with the same number. And suppose they agree on the first $n - 1$ terms, but differ at the nth term, like this:

$$a_1, a_2, a_3, \ldots, a_{n-1}, a_n, \ldots, a_N$$

$$a_1, a_2, a_3, \ldots, a_{n-1}, b_n, \ldots, b_N$$

where $a_n \neq b_n$, but $a_N = b_N$. Let's call the second sequence the "target" sequence.

Jasmine: As before, we know that a_{n-1}, b_n must occur consecutively in the first sequence somewhere after the nth term—that is, assuming that $a_{n-1} \neq a_n$, which we can assume without loss of generality. So the sequence goes

$$a_1, a_2, a_3, \ldots, a_{n-1}, a_n, \ldots, a_m = a_{n-1}, a_{m+1} = b_n, \ldots, a_N.$$

Emily: This is where we ran into the problem with the old splice. We needed to find an occurrence of a_n after the $(m + 1)$-th term so that we could splice the subsequence from the $(m + 1)$-th term to just before that following a_n in between the $(n - 1)$-th and nth terms. But your counterexample shows that there may not be such an occurrence of a_n.

Jasmine: With the swap operation, we have more flexibility. What we need is to find any one of the terms in positions $n - 1$ through m repeated after the $(m + 1)$-th term, for then we could perform a swap and get $a_{m+1} = b_n$ to appear right at position n.

Emily: What would force that to occur? Maybe we can assume that it does not occur and see if that leads us to some kind of impossibility?

Jasmine: Okay. It's funny, because I can see that the terms in positions $n + 1$ to m must occur after the nth position in the target sequence, which is b_n there.

Emily: How's that?

Jasmine: The two rational rompers represent the same rational numbers, and for any k strictly between n and m, our first sequence represents a_k/a_{k+1}, so our target sequence must represent that rational number as well. Now it can't represent that number as a ratio of consecutive terms among the first $n - 1$ terms because the two sequences agree there, and we're assuming that $a_{n-1} \neq a_n$; so it isn't equal to a_{n-1}/b_n, which is the ratio of the $(n - 1)$-th and nth term of the target sequence. Therefore, a_k, a_{k+1} must occur consecutively in the target sequence after its nth term.

Emily: Oh, nice! Actually, I think that argument also shows that a_{n-1} and a_n both appear after the nth term in the target sequence. In fact, they must appear consecutively.

Jasmine: You're right!

Emily: But how do we get these terms to appear after the $(m + 1)$-th term of the original sequence?

Emily and Jasmine think.

Jasmine: Do the two rational rompers have to have the same quantity of each number that appears in them? I'm wondering, because suppose that a_{n-1} does not occur in positions n through $m + 1$ in the original sequence. You just pointed out that a_{n-1} *must* occur after the nth position in the target sequence. If the two rational rompers must have the same number of occurrences of a_{n-1}, that would force an occurrence of a_{n-1} after the $(m + 1)$-th term of the original sequence! If that didn't occur, the original sequence would have fewer occurrences of a_{n-1} than the target sequence.

Emily: Intriguing! That sounds plausible. Let's see if we can show that. So forget about the notation we've set up so far and let's start anew with two finite rational rompers that represent the same rational numbers and have the same last term. Let's call them a_k and b_k, and let's suppose that there are N terms total. So we know both begin with 0, then 1, and $a_N = b_N$.

Jasmine: Suppose x appears n times in the a_k sequence. We want to show that x appears n times in the b_k sequence. If x is 0, we already know that 0 only appears once and at the very beginning of every rational romper, so let's assume $x > 0$. For the moment, let's assume that x is not a_N, so for both sequences every occurrence of x is in the middle of the sequence. And let's assume that x is not 1 either, so we also know that x is never followed by x (since x is not relatively prime to itself when $x > 1$). In that case, every occurrence of x will be preceded and followed by numbers that are not x, representing $2n$ rational numbers, with half having x in the numerator and half having x in the denominator (when expressed in lowest terms).

Emily: Since those rational numbers are also represented by the b_k sequence, x must appear exactly n times in that sequence, too!

Jasmine: Super! Now, what if x is the last term of the sequence $a_N = b_N$?

Emily: If $x > 1$, similar reasoning applies. For then the $n - 1$ occurrences before the last term will each be sandwiched between two numbers, neither of which equals x, so both sequences must represent $2(n - 1) + 1 = 2n - 1$ rational numbers that have an x in the numerator or denominator (when expressed in lowest terms).

Jasmine: So all we have to do is show that both sequences have the same number of 1s.

Emily: The potential hangup that we avoided was if 1, 1 ever occurs. If 1, 1 does occur, it means that the sequence represents 1; and if 1, 1 occurs in the a_k sequence, it must also occur in the b_k sequence, and vice versa. All other occurrences of 1 cannot be next to another occurrence of 1.

Jasmine: Actually, if 1, 1 does occur, since it must then occur in both sequences, we can just temporarily erase the first of these 1s to obtain rational rompers that still represent the same rational numbers and end with the same number. So it suffices to show that both sequences have the same number of 1s when 1, 1 does not occur.

Emily: Nice! So assume 1, 1 does not occur. Suppose there are n ones. Then I think that the number of rational numbers represented that have a 1 in the numerator or denominator (when written in lowest terms) is equal to $2n - 1$ or $2n$, depending on whether 1 is or is not the last term of the sequence, respectively. Either way, that means that both sequences must have the same number of 1s!

Emily: So it's true!

Jasmine: Yes, that's a nice fact to know.

Emily: Actually, must the two sequences begin and end with the same numbers? That is, suppose I write down two lists of N nonnegative integers and every pair of consecutive numbers that appear in one list also appears in the other. Do the two lists necessarily have to contain the same quantity of occurrences of each number?

Emily and Jasmine think.

Jasmine: Hmm. I'm afraid that fails. Consider 1, 2, 1 and 2, 1, 2.

Emily: How about that! Such a small counterexample!

Jasmine: In any case, let's see if we can perform swaps to transform one rational romper to another, now that we know that the two sequences must contain the same quantity of every number.

To be continued …

Learn by Doing

Random Numbers

by Addie Summer

Note: Problems vary in level of involvement. You'll find that you can do some quickly, but others (in red) are intended to give you something to think about for days. You're always welcome to email us or ask a mentor about this at the club.

Probability theory is the mathematical way of handling uncertainty. It's an important topic because life is full of uncertainty.

However, this Learn by Doing will be focused on a narrow subtopic of probability theory: random numbers. We assume some familiarity with probability.

Let's start with a familiar way in which random numbers are generated: die rolling.

1. A standard die is a cube with the numbers 1 through 6 written on its faces. When a die is rolled, it'll tumble and bounce, eventually landing with one of its sides facing up. The number on that top face is taken as the outcome of the die roll. Roll a die several times. Assuming that the die is a perfect cube and that the numbers on its faces do not affect the physical properties of the die in any way, what fraction of the tosses would you expect to come up 6?

By symmetry, there's no reason to expect any outcome of the die to be favored over any other outcome. Since there are 6 possible outcomes, we would expect each to occur on 1/6 of the rolls.

An actual die will not be a perfect cube, and any departure from perfect symmetry might be reflected in the frequency with which each number occurs. If a process produces numbers in such a way that all possible outcomes are equally likely, we say that the numbers are produced **uniformly at random**, or that the probability is uniformly distributed, or that the process is modeled by a **uniform probability distribution**.

If you actually role a perfect die several times, it's in fact not very likely that each number will come up the same number of times. We'll explore this more in a bit. For the moment, let's look at two ways of generating more numbers using dice. Assume that the dice are ideal dice which produce the numbers 1 through 6 according to a uniform probability distribution.

2. If you take two dice and roll them simultaneously and take the outcome to be the sum of the two numbers on the top faces of both dice, you will get a number between 2 and 12, inclusive. If you do this many, many times, for each number from 2 to 12, determine the fraction of rolls that you would expect to come up that number. In other words, for each possible outcome, what is its probability?

(As a check, the probability of getting a 10 is 1/12.)

3. Again, take two dice, with one colored red and the other blue. Roll them and let R be the outcome of the red die and B be the outcome of the blue die. Instead of taking the number $R + B$ as the outcome (as in Problem 2), we take the outcome to be $6(R - 1) + B$. What are the possible outcomes? For each outcome, what is its probability?

(Spoiler alert!) Problem 3 yielded another uniformly probability distribution. However, Problem 2 describes a process that produces random numbers with a nonuniform distribution.

4. Think of at least two more ways to generate random numbers and think about what the probability distribution is for each method.

5. If you were having trouble coming up with ways to generate random numbers, consider the following, and for each, describe its associated probability distribution as accurately as you can:

A. You create a spinner by dividing a circle into three parts by first dividing it in half with a diameter, and then drawing in a single radial line perpendicular to that diameter. You label the two smaller sectors 1 and 2 and the large sector 3. You tack a spinning needle onto the center of the circle and get random numbers by spinning the spinner, and after the spinner spins too many times to count, taking the number that it points to when it finally comes to rest.
B. You toss ten quarters high into the air. When they land and settle down, you count the number that show heads.
C. You throw a dart at a map of the United States and find the nearest city to wherever your dart hits the map. You look up the population of that city and take its leading digit. (Careful with this one—try it, if you can!)

The point of these first problems is to understand that different ways of generating random numbers have different probability distributions, and if we are to be able to predict outcomes well, we need to know the probability distribution.

7. Role a perfect die 6 times to get 6 random numbers. Although the probability of each outcome is 1/6, it's not very likely that each of the numbers 1 through 6 comes up when you do this. What is the probability that each number comes up once? What is the probability that this does *not* happen (i.e., that fewer than 6 different numbers come up)?

8. In the setup of Problem 7, for each n from 1 to 6, compute the probability that 6 comes up n time(s).

9. In the setup of Problem 7, for each n from 1 to 6, compute the probability that exactly n different number(s) come up.

Things get more interesting when there are infinitely many possible outcomes!

10. Explain why it is impossible to pick a random positive integer with a uniform probability distribution.

11. Let's generate a random positive integer in the following way: We flip a coin over and over until it comes up heads. The number of times we had to flip the coin is our random positive integer. What is the associated probability distribution?

12. Suppose a method for produce random positive integers has the property that the probability that n occur is 1.01 times the probability that $n + 1$ occurs. What is the probability that 1 occurs? Can you think of a real-world way to generating positive integers that produces random positive integers according to this distribution, at least in theory?

13. Suppose you have a method for generating random positive integers with the property that the probability that the outcome is greater than n is equal to $1/(1 + n)$. What is the probability

that n is the outcome? Can you think of a real-world way to generating positive integers that produces random positive integers according to this distribution, at least in theory?

To specify a random number, all we need to specify is its probability distribution. We do not need to know the mechanism by which the random number is generated. For a random number generator that produces positive integers, that would mean specifying the probability p_n that the outcome is n, for each positive integer n. The only conditions that we require of the p_n are that they are nonnegative numbers which add up to 1.

Now let's consider a random number generator that produces a real number between 0 and 1. We will produce this random number by specifying its binary representation. Since the number is between 0 and 1, the only nonzero binary digits occur after the binary point. For each place after the binary point, we assign a fair coin. Since there are infinitely many places after the binary point, we have infinitely many coins. We imagine flipping all of them at the same time. If a coin comes up tails, we place a 0 in the associated place in the binary number, otherwise we place a 1 there.

We will use this method of producing random real numbers for Problems 14-17.

14. Let n be a real number between 0 and 1. Show that the described method of producing a random real number between 0 and 1 produces the number n with probability 0.

Unlike our method for producing random positive integers, the probability of getting a specific number is 0. How can that be? How can we have something that produces a random number, yet the probability that any particular number come up be 0?

15. While the probability of generating a particular number is 0, show that the probability that the outcome is between a and b, where $0 \le a < b \le 1$ is $b - a$.

Because of Problem 15, we call the distribution associated to this random number generator the **uniform probability distribution on the interval [0, 1]**.

16. Show that the probability of producing a rational number between 0 and 1 is 0.

17. Suppose we use this random number generator twice and take the sum of the results. The sum will be a real number between 0 and 2. If $0 \le a < b \le 2$, what is the probability that the sum is between a and b?

18. Instead of using coin tosses to determine the binary digits of a real number, suppose you use a spinner that comes up 0 with probability p and 1 with probability q. To each binary place after the binary point, you assign such a spinner. To get the random real number, all the spinners are spun simultaneously. What is the probability that the real number generated is between $k/2^m$ and $(k + 1)/2^m$, where k is an integer such that $0 \le k < 2^m$? What is the probability that the real number generated is between 1/3 and 1/2? What is the probability that the real number generated is between 1/2 and 2/3?

19. Can you think of a way to generate a random real number where every real number is a theoretical possible outcome (even though nearly all real numbers will come up with probability 0)? What is the probability distribution of your scheme?

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Girls' Angle

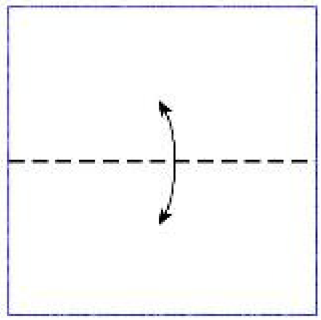
Origami Pentagon

by C. Kenneth Fan

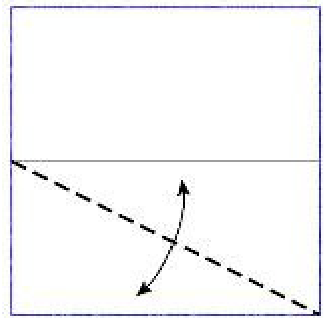
Some members at the club challenged themselves to figure out how to fold a regular octagon with a piece of origami paper. It's a wonderful project and a natural follow-up question is, "What regular polygons can be folded using origami techniques, and how can they be folded?" Here's one way to fold a regular pentagon. Can you explain why this fold sequence works? Can you come up with a better way?

Before you begin, try to figure out a way to fold an origami regular pentagon yourself.

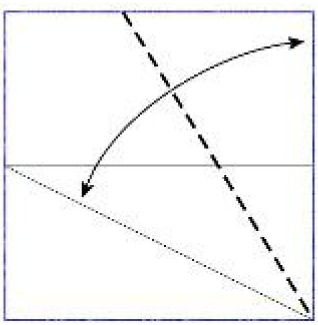
Even if you don't come up with a way, trying yourself helps to get your mind familiar with the issues involved.



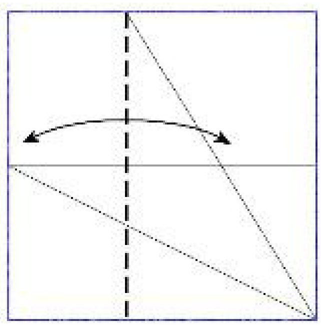
1. Begin colored side down. Fold in half. Unfold.



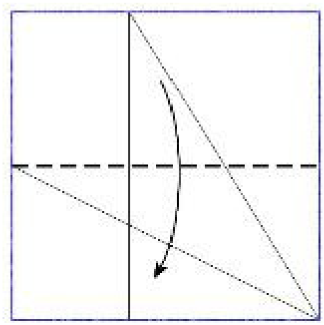
2. Fold and unfold



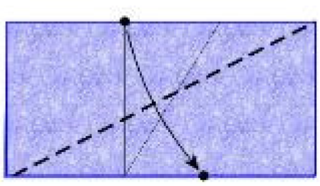
3. Fold the right edge in to lie along the crease you just made. Unfold.



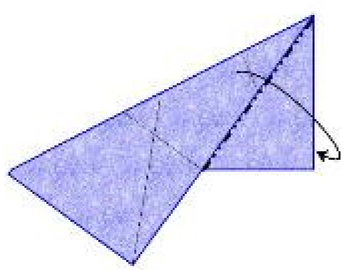
4. Fold the left edge in along a vertical crease that intersects the upper edge at the same place where the crease you just made does.



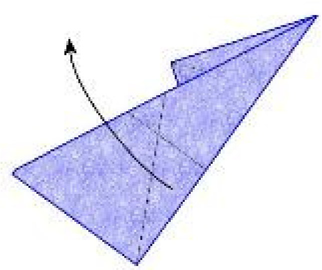
5. Fold in half along the existing crease.



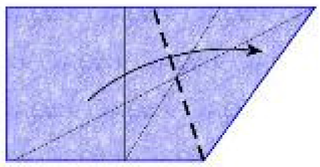
6. Make sure your paper is oriented as in the diagram with the raw edges at the bottom, not the top. Fold the indicated point onto the lower edge along a crease that goes through the upper right corner. Note that this crease falls short of the lower left corner.



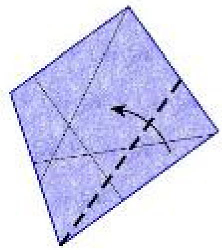
7. Fold the right side around along a crease that lines up with the right edge of the near layer.



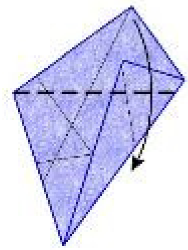
8. Unfold the near layer.



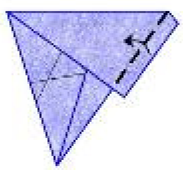
9. Fold the right side over so that the crease lies along the edge of the hidden layer beneath. This crease should pass directly through the lower right corner and the intersection of the two diagonal creases that already exist.



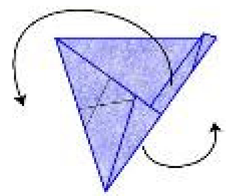
10. Fold both layers along a crease that runs along the edge of the hidden layer beneath. The lower edge should meet the intersection of two creases that already exist.



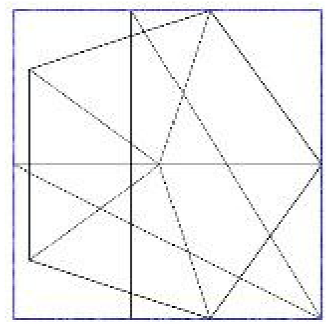
11. (Not all creases are shown.) Fold down through all layers along a crease that runs along the edge of the hidden layer beneath.



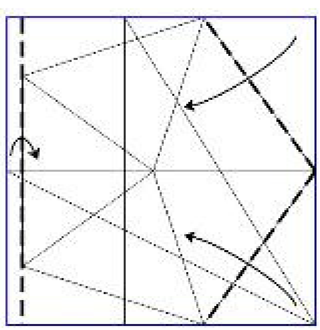
12. (Not all creases are shown.) Fold the excess strip along a crease that runs along the edge of the hidden layer beneath. Note that the long edge of the excess strip should be parallel to the crease.



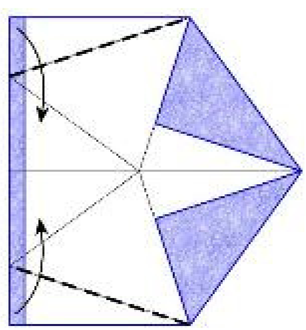
13. (Not all creases are shown.) Gently unfold back to the original square.



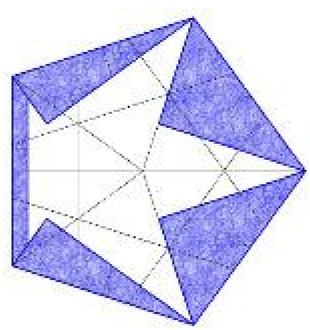
14. Only important creases will be shown from now on. You should be able to find a large regular pentagon creased into your square.



15. Fold in along three creases that run through three of the five sides of the pentagon. The fold on the left should be vertical.



16. Fold in the upper and lower left corners along two more creases that run along sides of the pentagon.



17. Finished Regular Pentagon!

What are the measures of the interior angles of a regular pentagon? If the side length of the original square is s, what is the side length of the resulting pentagon? Is this the largest regular pentagon that can be folded from an origami square of given size?

Notes from the Club

These notes cover some of what happened at Girls' Angle meets. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

Session 33 - Meet 8 November 2, 2023	Mentors:	Jade Buckwalter, Anushree Gupta, Gautami Mudaliar, Hanna Mularczyk, AnaMaria Perez, Padmasini Venkat, Jane Wang, Doris Woodruff, Saba Zerefa, Angelina Zhang, Jasmine Zou

What are the volumes of each of the Platonic solids as a function of the length of one of its edges?

Session 33 - Meet 9 November 9, 2023	Mentors:	Elisabeth Bullock, Jade Buckwalter, Anushree Gupta, Gautami Mudaliar, Hanna Mularczyk, Tharini Padmagarisan, AnaMaria Perez, Vievie Romanelli, Swathi Senthil, Padmasini Venkat, Jing Wang, Dora Woodruff, Saba Zerefa, Angelina Zhang

Can you devise a way to fold a regular octagon using origami techniques? For a regular pentagon, see page 26.

Session 33 - Meet 10 November 16, 2023	Mentors:	Elisabeth Bullock, Jade Buckwalter, Shauna Kwag, Gautami Mudaliar, Hanna Mularczyk, Tharini Padmagarisan, AnaMaria Perez, Vievie Romanelli, Swathi Senthil, Padmasini Venkat, Jane Wang, Jing Wang, Dora Woodruff, Saba Zerefa, Angelina Zhang

What's the most efficient algorithm you can come up with to factor a given integer?

Session 33 - Meet 11 November 30, 2023	Mentors:	Anushree Gupta, Shauna Kwag, Gautami Mudaliar, Hanna Mularczyk, Tharini Padmagarisan, AnaMaria Perez, Saba Zerefa, Angelina Zhang

What's the general equation for an ellipse in the plane (whose major and minor axes are not necessarily aligned with the coordinate axes)?

Can you invent your own, personal, way of deriving the quadratic formula?

Session 33 - Meet 12 December 7, 2023	Mentors:	Elisabeth Bullock, Jade Buckwalter, Anushree Gupta, Hanna Mularczyk, Swathi Senthil, Jane Wang, Jing Wang, Dora Woodruff, Saba Zerefa, Angelina Zhang

Finite dimensional vector spaces over finite fields have finitely many points. How many k-dimensional subspaces are there of an n-dimensional vector space over a finite field with q elements?

Calendar

Session 33: (all dates in 2023)

September	14	Start of the thirty-third session!
	21	
	28	Support Network Visitor: Isable Vogt, Brown University
October	5	
	12	
	19	
	26	
November	2	
	9	
	16	
	23	Thanksgiving - No meet
	30	
December	7	

Session 34: (all dates in 2024)

February	1	Start of the thirty-fourth session!
	8	
	15	
	22	No meet
	29	
March	7	
	14	
	21	
	28	No meet
April	4	
	11	
	18	No meet
	25	
May	2	
	9	

Girls' Angle has run over 150 Math Collaborations. Math Collaborations are fun, fully collaborative, math events that can be adapted to a variety of group sizes and skill levels. We now have versions where all can participate remotely. We have now run four such "all-virtual" Math Collaboration. If interested, contact us at girlsangle@gmail.com. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

Alphanumerics 3G: $5^2 \cdot 13 \cdot 73 \cdot 6561193 \cdot 42139162737709816956364437$ (base 10) or $E^2 \cdot M \cdot LIIGD \cdot AYVITCNSZKRGRCQZAK$ (base 27).
Alphanumerics 4G: $13 \cdot 457 \cdot 1753 \cdot 157461170888808787579472095 3687$ (base 10) or N·KB·YQB·ZQXCLQRUøXGOZøFQLUEIIN (base 27).
Here, ø stands for the digit zero in base 27.

Girls' Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ____________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

____-

The $50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- ☐ Enclosed is a check for $50 for a 1-year Girls' Angle Membership.
- ☐ I am making a tax-free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

Girls' Angle

A Math Club for Girls

Girls' Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is $20/meet for members and $30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle is based in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, founder and director of the Exploratory
Yaim Cooper, Institute for Advanced Study
Julia Elisenda Grigsby, professor of mathematics, Boston College
Kay Kirkpatrick, associate professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, assistant dean and director teaching & learning, Stanford University
Lauren McGough, postdoctoral fellow, University of Chicago
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, associate professor, University of Utah School of Medicine
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Liz Simon, graduate student, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, associate professor, University of Washington
Karen Willcox, Director, Oden Institute for Computational Engineering and Sciences, UT Austin
Lauren Williams, professor of mathematics, Harvard University

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high-level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: ________________________

(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is $216 for one session (12 meets)	☐ I will pay on a per meet basis at $30/meet.
☐ I will pay on a per meet basis at $20/meet.	☐ I'm including $50 to become a member, and I have selected an item from the left.

☐ I am making a tax-free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: ________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____________________________________