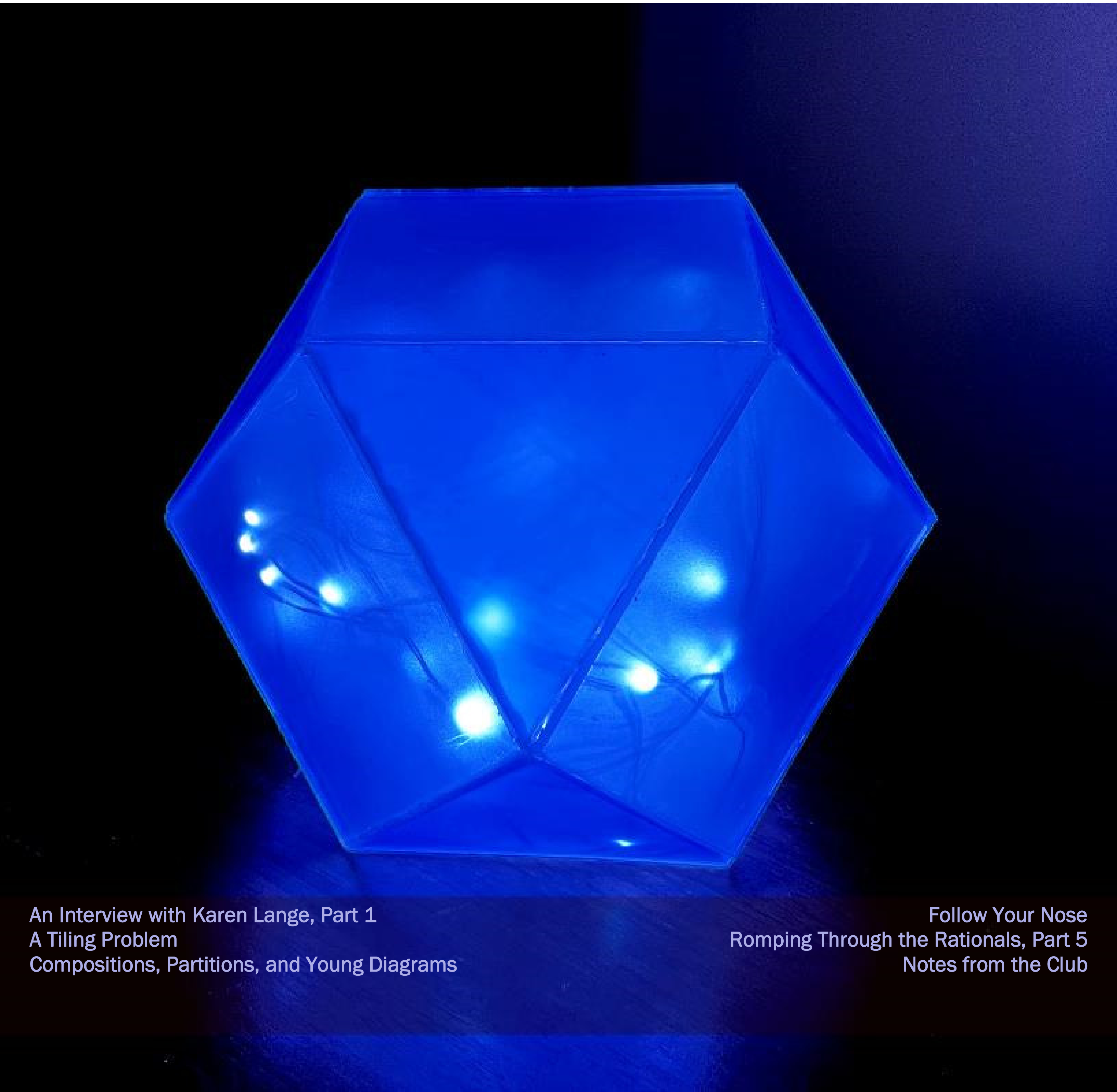
Girls' Angle Bulletin

October/November 2023 • Volume 17 • Number 1

To Foster and Nurture Girls' Interest in Mathematics



From the Founder

The presentation of math bears little resemblance to the way mathematics is actually done. Behind that nifty theorem and proof you read, there's a rich history of precursor ideas, computations, confusions, and realizations – fun and excitement rarely mentioned. - Ken Fan, President and Founder

Girls' Angle Donors

A Heartfelt Thank You to our Donors!

Individuals

Uma Achutha
Nancy Blachman and David desJardins, founders of the Julia Robinson Mathematics Festival, jrmf.org.
Michael Beckman
Ravi Boppana
Marla Capozzi
Yungyuan Chang
Suhas Daftuar
Patricia Davidson
Ingrid Daubechies
Anda Degeratu
Ellen Eischen
Glenn and Sara Ellison
John Engstrom
Lena Gan
Jacqueline Garrahan
Courtney Gibbons
Shayne Gilbert
Vanessa Gould
Rishi Gupta
Larry Guth
Andrea Hawksley
Scott Hilton
Delia Cheung Hom and Eugene Shih
David Kelly
Brian and Darlene Matthews
Toshia McCabe
Mary O'Keefe
Stephen Knight and Elizabeth Quattrocki Knight
Sarah Manchester
Alison and Catherine Miller
Beth O'Sullivan
Robert Penny and Elizabeth Tyler
Malcolm Quinn
Jeffrey and Eve Rittenberg
Marcine Snyder
Eugene Sorets
The Waldman and Romanelli Family
Marion Walter
Andrew Watson and Ritu Thamman
Brandy Wiegers
Brian Wilson and Annette Sassi
Lissa Winstanley
Cunyuan Zhang
The Zimmerman family
Anonymous

Nonprofit Organizations

Draper Laboratories
The Mathenaeum Foundation
Orlando Math Circle

Corporate Donors

Adobe
Akamai Technologies
Big George Ventures
D. E. Shaw
John Hancock
Maplesoft
Massachusetts Innovation & Technology Exchange (MITX)
Mathenaeum
MathWorks, Inc.
Microsoft
Microsoft Research
Nature America, Inc.
Oracle

For Bulletin Sponsors, please visit girlsangle.org.

Girls' Angle Bulletin

The official magazine of
Girls' Angle: A Math Club for girls
Electronic Version (ISSN 2151-5743)

Website: www.girlsangle.org
Email: girlsangle@gmail.com

This magazine is published six times a year by Girls' Angle to communicate with its members and to share ideas and information about mathematics.

Girls' Angle welcomes submissions that pertain to mathematics.

The print version of the Bulletin is printed by the American Mathematical Society.

Editors: Amanda Galtman
Jennifer Sidney
Executive Editor: C. Kenneth Fan

Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: A cuboctahedral lamp designed and created by Sofia Egan. Sofia attends the Buckingham, Browne, and Nichols Upper School in Cambridge, MA.

An Interview with Karen Lange, Part 1

Karen Lange is the Theresa Mall Mullarkey Associate Professor of Mathematics at Wellesley College. She earned her Bachelor of Arts in Mathematics with a minor in Computer Science with High Honors from Swarthmore College. She then earned a PhD in mathematics at the University of Chicago under the supervision of Robert Soare and Denis Hirschfeldt.

Ken: What's the first mathematical idea that caught your interest? What did you find interesting about it, and about how old were you?

Karen: I really love this question, and two things came to mind. One was in sixth grade, and we were learning about the distributive property of addition. There was something where we were supposed to always use the distributive property to solve something in addition: $D(A + B)$ is equal to $DA + DB$.

And I didn't use the distributive property the way that I was supposed to in this entire assignment. I remember being really frustrated, because even though I was correct, the entire thing was marked as wrong because I hadn't done what I was supposed to do. And I remember my dad trying to explain to me what I was supposed to do, and me just being very confused, because I thought, "But this was true."

I was supposed to write A plus B, and then multiply by D, and I had probably just done the opposite. I remember that being very frustrating, this idea that something can be true, but not what a teacher is looking for. That I was having trouble seeing the difference between what

Once you realize that you can make discoveries yourself, well, then you're unstoppable, because you have a better understanding of where any math that you've learned comes from.

they wanted me to do, versus just solving the addition problem.

That's not the best example, though. My other example actually was an idea that I kind of came up with, like my first mathematical discovery that was mine. Obviously, it wasn't new or anything like that. I think it was in ninth grade, and a teacher had given us a bunch of Pythagorean theorem kinds of problems about distance. They were just, "Oh, find the distance between these two points in the plane."

And he gave us a bunch of these, and I remember thinking, "Wait a minute, I'm just using the Pythagorean theorem every single time to solve these distance-between-two-points problems." I should say, he hadn't told us about the existence of a formula for the distance between two points in a line. And so, I remember being in my bedroom, thinking, "I'm doing the same thing every single time. I could write a formula for this. There's a formula here."

I got really excited that I came up with this formula, (x_1 minus x_0) squared plus (y_1 minus y_0) squared, and then the square root. I didn't think it was by itself all that interesting, but I was a little bit like, "We're doing the same process over and over again. My teacher should know that there's a formula for this." I remember going to the teacher the next day, and saying, "We're just doing this formula over and over again. Why are we doing this?" Now, as a teacher, I know why he was having me do it.

But I remember him telling me, "Oh, you just read it in the book." And I remember being very offended. I was like, "What do you mean it's in the book? I just figured this out." I remember, in general, being excited about realizing I could create mathematics, and then again, being a little bit frustrated that he was thinking I read it in the book. Embarrassingly at the time, I was not reading the book much at all.

But anyway, that was probably somewhere in ninth grade, and I really appreciate that he — Mr. Gustavson was my teacher — set up a lot of experiences like that that were discovery things.

Ken: That is a great story! I think that is one of the conceptual breakthroughs that has to happen in the whole long process of becoming a mathematician. That's pretty cool that you had the idea on your own there, just coming to you as an idea, while you were working out these distance problems. Did that experience affect the way you teach math today?

Karen: Definitely. Actually, the reason I remember Mr. Gustavson — the other thing about his class that was amazing — is that it was a class called "Algebra, Logic and Proof." Everybody had seen algebra before, but seen just the manipulations, and the idea of the class was to introduce us all to the idea of proof and logic.

So, we would study truth tables, and then we would use a very simple proof system to prove a lot of the algebra that we learned. We had learned a lot of the rules of algebra previously, but then how do you start from essentially the properties of real numbers, and of integers, depending on the context, and derive all of these basic algebraic facts that we use?

And so, on one side, there was a lot of skill-building where you could learn by doing the rote work, but then it connected to what we were doing. On the discovery side, I realized, as you say, the agency: "Oh, I can discover some mathematics."

On that side, it's definitely influenced my teaching in the sense that I want to try to give students that experience. So, I love to teach. Sometimes I get to teach a first-year number theory class, and I like to teach it in the discovery-based, inquiry-based learning style of, "Hey, work out some examples. Write out the first 100 primes or at least the primes within 100, and look for patterns there. Once you have your patterns, are there things that you can prove?"

Because I do find there's just so much power in ownership, in knowing that you discovered it yourself. Other people have discovered it before, but once you realize that you can make discoveries yourself, well, then you're unstoppable, because you have a better understanding of where any math that you've learned comes from. You understand where it came from, but then you also realize, "Oh, wait. I can start asking my own questions, seeing my own patterns, and then trying to prove them."

So, definitely, it impacts how I teach in terms of trying to find ways, within the constraints of what content I'm "supposed" to be teaching, to give students some of that discovery experience. The other part, though, that I was alluding to with this class was that it was my first introduction to proof — this idea that I could prove things. I could know and be sure of what was true, without needing external authority. There is this idea that I should be able to know and determine if something is true, just using my own logic and capabilities, and that I can come up with a written verification of what's true, on my own. So, that class just blew my whole mind.

Ken: At that age, were you already aware of the difference between belief and proof? A lot of students know things, and they'll say them as if they are definitely true, and they will simply think that the statements are definitely true and don't need proof.

Karen: I'm not sure if I, until that class, had made that realization. I think, unfortunately, it wasn't necessarily that I believed. I think that my philosophy was more like, "The authorities have told me the truth, and I believe the authorities, right?" It was a belief in authority, and realizing that with mathematics, I can be the authority for myself. I didn't need somebody higher up with more expertise to tell me what was true or false. I, myself, could be the determiner of truth, which again goes to that agency part of mathematics.

Ken: That's really cool. When you're using this style of teaching, do you sometimes have students who have difficulty articulating what they notice? How do you help those students?

Karen: That's a great question, because that's one of the big challenges in teaching, articulating what you notice. One thing that I find really helpful about some of these discovery styles of teaching, is actually for me to back up, and — like your club — have people, students, peers doing math together. Oftentimes, a reason people aren't so good at articulating patterns they see is a fear that anything they think they see "isn't right." Sometimes, they don't think they see anything, but sometimes it's, "Well, I see this, but I'm not confident that that's the right thing to see." Then, people just won't even put ideas out there.

The peer aspect is so helpful when I back out of the situation and people are working with other people who are going through that same experience of discovering their own agency. They see, "Oh, another student isn't so sure either. I notice this seems to be true. I don't know if this is right, but maybe it's something — there's something going on with all the even numbers doing something," and the other student realizes, "Oh, I thought there might be something with the even numbers, but I didn't have the confidence yet to say it."

> *"What do you mean it's in the book? I just figured this out."*

When they see other students who are maybe a little bit braver put themselves out there, that actually goes a long way. There's also the scale of interaction, having people crowdsource ideas, maybe not in front of me, but just having two or three people brainstorm their ideas together.

At first, the student might not articulate what their ideas are, but after they get more practice with smaller groups, and especially smaller groups of peers who are in the same place in their math journeys, then they get braver. They realize, "Oh, I'm not the only one who wasn't sure but also thought there was something about the even numbers having some kind of property."

So, actually, I've had to learn to get more out of the way. It's less about me directly interacting than about scaffolding these smaller group peer-to-peer interactions. I've helped bring people out, but this is a lifelong journey.

To be continued…

A Tiling Problem

by Emily Caputo, Sophie Harteveldt, and Alina Patwari
edited by Amanda Galtman

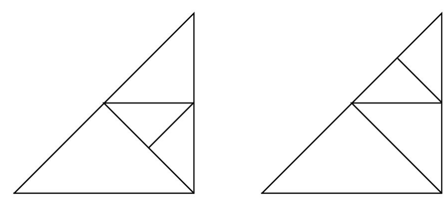
Primary mentorship for this mathematical investigation was provided by Elisabeth Bullock, Ken Fan, and Swathi Senthil. We believe this is a novel tiling problem.

Introduction

Do you have any interest in triangles? If so, continue reading. Our journey with triangles began when the second author pondered this problem: What type of triangle can be split into three congruent triangles, each similar to the original? In thinking about this problem, she came up with an apparently novel tiling problem: What is the number of ways one can dissect a right isosceles triangle into smaller right isosceles triangles?

We call any such dissection into N tiles an ***N*-tiling**. Working on this problem manually proved to be challenging, because the number of N-tilings grows extremely fast with N and it was hard to think of a systematic way to find them all. Even for $N = 5$, there are many ways to dissect into five tiles. To limit the number of tilings, we added another requirement: All the tiles must be edge-to-edge. In other words: Any overlapping edges must touch at all points of both edges. The "edge-to-edge" requirement was inspired by Johannes Kepler's work in *Harmony of Worlds*, where he explores different subsets of edge-to-edge tilings.

With this reformulation, we found an efficient algorithm for finding all finished tilings. The algorithm also provides an upper bound on the number of N-tilings.



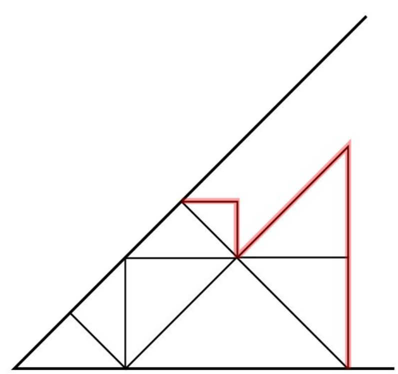
A non-edge-to-edge 4-tiling (left) and an edge-to-edge 4-tiling (right).

Definitions and Terminology

Tile: An isosceles right triangle.

Finished Tiling: An edge-to-edge tiling of an isosceles right triangle with isosceles right triangles. See the bottom of page 10 for examples.

Unfinished Tiling: An edge-to-edge tiling consisting of tiles whose union is not, itself, an isosceles right triangle but fits into the corner of a 45° angle. The union of all the tiles plus the sides of the 45° angle must not have any holes.



An unfinished tiling with its frontier indicated in red.

We use **tiling** to refer to either a finished or unfinished tiling.

Frontier: When a tiling is placed into the corner of a 45° angle so that the union of all the tiles and the sides of the 45° angle have no holes, the frontier is the boundary of the tiling that connects the two sides of the 45° angle and separates the interior of the 45° angle into a part completely covered by tiles and a part completely devoid of tiles.

***N*-tiling**: A tiling consisting of N tiles.

The Tiling Problem

We orient an isosceles right triangle so that a Cartesian coordinate system could be placed upon the triangle so that its vertices are located at (0, 0), (1, 0), and (1, 1). How many distinct N-tilings are there of this isosceles right triangle?

We count tilings that are mirror images of each other about the altitude to the hypotenuse as different tilings.

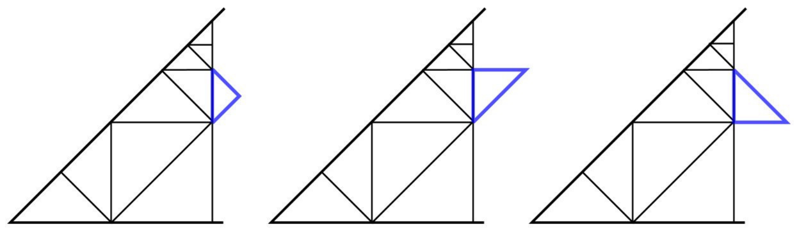
Development of Our Algorithm

To solve this problem, we originally created a "brute force" method for finding every possible finished tiling. The algorithm builds tilings by filling tiles into the corner of a big 45° angle, which we orient so that one side of the angle is the positive horizontal axis and the other side runs through the first quadrant. The algorithm is seeded with the two 1-tilings: one tiling with the tile's right angle vertex on the horizontal axis and one tiling with the right angle vertex on the side running through the first quadrant. The algorithm inductively finds every (N + 1)-tiling by adding one tile to every N-tiling in every possible way, staying within the bounds of the big 45° angle. The algorithm checks each of these tilings to see if its frontier is a vertical line segment connecting the sides of the 45° angle. If so, and if this particular finished tiling has not been found before, the algorithm adds the tiling to a list of finished (N + 1)-tilings. This method was logically sound but inefficient. The algorithm took a week to find the number of 10-tilings alone, and the amount of time it took to find N-tilings seemed to grow more than exponentially with N.

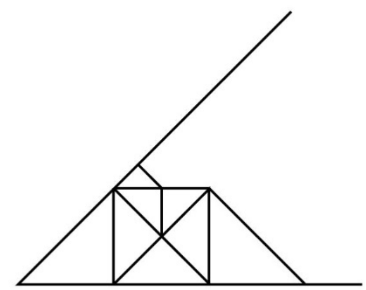
To make a more efficient algorithm, we created a system of codifying tilings by systematically adding tiles. The codification system is called the **lexicographic code**.

Here's how the code works. As noted above, there are only two ways to place the first tile into the corner of the 45° angle. The first way adds a vertical line in the big angle, and the second one adds a line that makes a 135° angle to the horizontal side of the big angle. (By accounting for both of these cases, we can be sure all finished tilings can be detected by checking if the frontier is vertical. That is, a finished tiling may be fit into the big 45° angle so that its frontier is slanted instead of vertical, and it won't be recognized as finished if we only check for a vertical frontier, but if it is flipped around the angle bisector of the big 45° angle, its frontier will be flipped to vertical, and that will be recognized as finished.) The first tile is coded as either 90 (for the vertical frontier case) or 135 (otherwise).

Each following tile is assigned a lexicographic code of the form (a, b), where a is a positive number and b is one of the three letters H, R, or L. The value a is the number of first edge on the frontier where this tile is attached, counting the edges on the frontier from bottom to top. (The frontier is composed of a bunch of line segments that are the edges of the tiles that run along it.) The value b codifies the orientation of the added tile. There are three possible orientations, which we label H, R, and L. These labels correspond to the location of the right angle of the added tile, relative to the frontier edge where the tile attaches. If the right angle vertex is not on the edge, that orientation corresponds to H. If the right angle vertex is on the edge's endpoint that is encountered first as you travel along the frontier from bottom to top, that orientation corresponds to L. The only remaining orientation corresponds to R.



For example, the figure above adds a blue tile to the frontier in all three ways (H, R, and L) from left to right. Since that tile attaches to the second edge from the bottom along the frontier, the lexicographic codes for the blue tile are (2, H), (2, R), and (2, L), respectively.



In this way, we codified each tiling as a list of lexicographic codes in the order that the tiles are added. To make this code unique, we adopted the convention of adding tiles in such a way that the *a* value for each tile must be greater than or equal to the *a* value for the tile preceding it. For example, the tiling shown at right has the code

90, (1,H), (1,R), (1,R), (1,L), (2,H), (3,L), (3,H).

This method of adding tiles guarantees that each new finished tiling found is unique. The algorithm doesn't need to check whether the tiling has been found before, which saves a considerable amount of computation time. However, as N increased, we found that the algorithm allowed for holes to appear. This was not an issue in terms of efficiency, but it meant a working algorithm would be very hard to code. Therefore, we abandoned this method and tried to come up with a new ordering of the placement of tiles to guarantee an absence of holes.

Current Algorithm

To find finished tilings, as before, we start with a 45° angle situated in the Cartesian plane as before and fill this angle with tiles until they form a tiling with a vertical frontier. Our algorithm is a method of adding tiles that is efficient, finds all possible finished tilings, and does not find the same finished tiling more than once.

The idea is to systematically add tiles to the specific edge on the frontier that is first encountered when travelling along the frontier from the bottom up, among those frontier edges with an endpoint furthest to the left.

We again seed our search with the two ways to place a single tile in the corner of the 45° angle, and then proceed inductively on the number of tiles in the tiling.

The frontier consists of line segments that are the edges of tiles attached to the frontier. We keep track of the coordinates of the endpoints of each edge. To add our next tile, we find the endpoints with the smallest x-coordinate. If there are multiple vertices with the same minimum x-coordinate, we pick the one with the lowest y-coordinate. If this endpoint is shared by two tile edges on the frontier, we pick the edge we first encounter when traversing the frontier from the bottom. We call this segment the lowest, leftmost edge on the frontier. It is uniquely specified.

We then add a tile to this edge. While we have abandoned our lexicographic code, we still use the three orientations R, L, and H explained above to indicate how the tile is added. We create new, unique tilings by adding a new tile in each of the three orientations as long as adding the new tile stays inside the big 45° angle and doesn't overlap existing tiles.

If a new tiling has a vertical frontier, we add it to a list of finished tilings.

We repeat this process of adding a new tile to all ($N + 1$)-tilings found (including finished ones)!

Proof that the Algorithm Works

First, we prove that the algorithm never double-counts a finished tiling.

To prove this, consider a finished tiling that the algorithm finds twice. As the 45° angle is filled in with tiles, there must be two different ways of adding tiles (consistent with our algorithm) that produce this finished tiling. The first tile must be the same, but at some point, the paths to produce this finished tiling must diverge. Just prior to the moment of divergence, the in-progress tilings are identical and have the same frontier. Since the frontier uniquely determines the edge to which we add the next tile, the paths can diverge only due to a different orientation of the added tile. But then the finished tilings could not be identical.

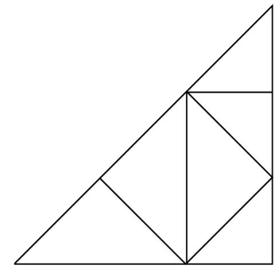
Next, we prove that every finished tiling is found by the algorithm.

To prove this, suppose we have a finished N-tiling T. If $N = 1$, there is only one finished 1-tiling. Since we seed the algorithm with that tile, the algorithm does find this case. So, assume $N > 1$. We can place this tiling into the big angle so its frontier is vertical. Starting from the tile in the very corner of the big angle, we shall prove by induction on the number of tiles added that the algorithm adds tiles until it reconstructs T. This corner tile creates a frontier to which another tile attaches, and the algorithm adds this other tile because the algorithm adds tiles in every possible orientation.

Suppose K of the N tiles in T have been added by the algorithm and that $1 < K < N$. There is a unique lowest, leftmost edge E on the frontier. If there isn't a tile in T that can be added to E, then E must be on the boundary of T, since there cannot be holes in the tiling. By algorithm design, no part of the frontier is to the left of E's leftmost endpoint. On the other hand, there cannot be tiles to the right of E's leftmost endpoint because if there were, the frontier of T would be a vertical line entirely to the right of E's leftmost endpoint. Therefore, the frontier of the tiling formed by the first K tiles must be a vertical line. However, because $K < N$, there must be tiles in T to the right of the vertical frontier formed by the first K tiles, and since the frontier of T is vertical, there must be a tile attached to E. By design, the algorithm will add that tile.

Upper Bound

Let $f(N)$ be the number of finished N-tilings. Our algorithm uniquely specifies which edge a tile is added to when constructing tilings. Therefore, we can specify each tiling uniquely by stating the orientation of the first, corner tile, and then giving a string of H's, L's, and R's. Thus, we have an immediate upper bound on the number of finished N-tilings: $f(N) \leq 2(3^{N-1})$. However, this upper bound includes many tilings that are not finished and ignores the fact that not every word in H, L, and R corresponds to a tiling. For example, some words require a tile outside the big 45° angle and some cause tiles to overlap. We expect that this upper bound can be substantially improved.



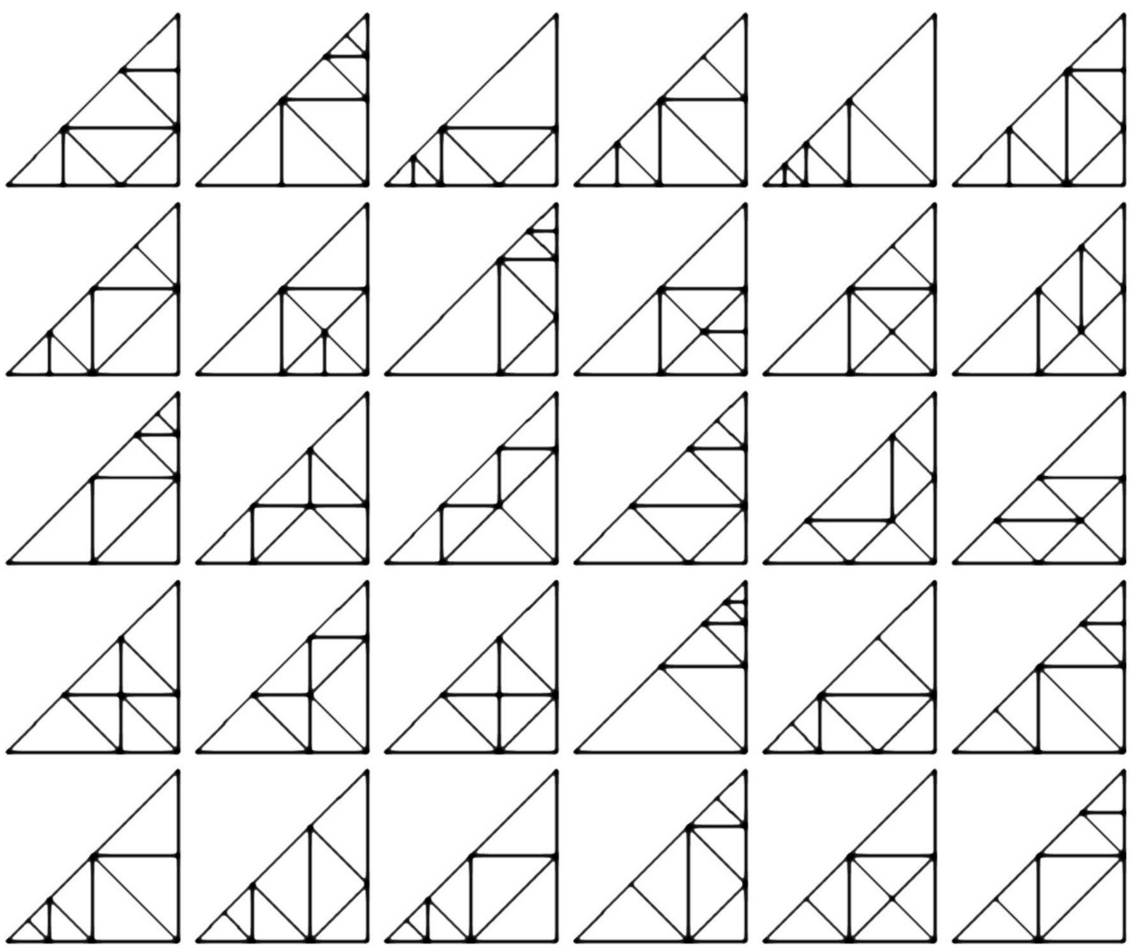
135 R H H H L

Numerical Results

This table shows what the algorithm computes for the values of $f(N)$ for $1 \leq N \leq 20$.

N	1	2	3	4	5	6	7	8	9	10
***f*(*N*)**	1	1	2	6	10	15	30	69	137	243
N	11	12	13	14	15	16	17	18	19	20
***f*(*N*)**	451	853	1562	2912	5555	10459	19644	37039	69305	129346

All 30 Edge-To-Edge 7-Tilings



Compositions, Partitions, and Young Diagrams

by Robert Donley[1]
edited by Amanda Galtman

We continue the development of partitions from the previous two installments. In this part, we assign to partitions a visual design that complements the generating function approach and that reveals properties not necessarily obvious in a purely numerical description of partitions. We keep the definitions and notation from previous installments; in particular, recall that partitions are denoted by non-increasing strings of digits and that $p_{[n]}(k)$ denotes the number of partitions of k with parts less than or equal to n.

First, we draw some direct connections between partitions and compositions. For a partition with largest part bounded by m, we assign a weak composition with m parts by recording the multiplicity of each part. For instance, the partition 554322 of $k = 21$ yields the weak composition 02112. To pass from a weak composition to the corresponding partition, we list the parts as many times as the composition indicates.

Exercise: What does the sum of the composition's parts equal?

Exercise: For all weak compositions of $k = 4$ with 3 parts, list the corresponding partitions.

Exercise: What type of partitions correspond to such compositions with binary parts? That is, we use parts equal to 0 or 1. Find the partitions corresponding to the binary numbers with four digits.

Another connection between partitions and weak compositions with a fixed number of parts follows from successive differences and partial sums. For instance, if we take successive differences of the partition 75433 of $k = 22$, we obtain the composition 21103. Here the last parts of the composition and partition coincide, as the final difference subtracts 0. To reverse this process, we take partial sums from the right, such as $3 + 0 + 1 = 4$.

Exercise: For a weak composition obtained by taking successive differences of the parts of a partition, what does the sum of the composition's parts equal?

Exercise: Find the corresponding weak compositions for the partitions of 5.

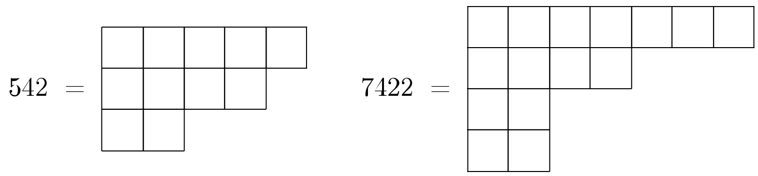
Exercise: Which partitions give rise to weak compositions (via successive differences) such that the last part is the only nonzero part? To weak compositions with binary entries and, in particular, with all entries equal to 1?

Compositions and partitions diverge with respect to general techniques. Since a rearrangement of a composition yields a new composition, compositions display a large degree of symmetry, which points to the framework of **group theory**. On the other hand, there is a powerful visual approach to partitions that reveals an important symmetry not available to compositions.

[1] This content is supported in part by a grant from MathWorks.

Definition: A **Young diagram** for a partition is an arrangement of rows of squares, justified to the left. Each part corresponds to a row with that many squares, and the rows are listed from the top in non-increasing order by number of squares. Partly to make generating functions nicer to express later, we consider the diagram with no squares to be a Young diagram.

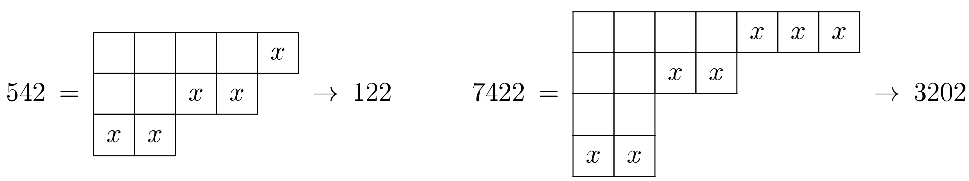
For example, the corresponding Young diagrams for 542 and 7422 are



Exercise: What Young diagram results from a partition with a single part? From a partition with all parts equal to 1? From a partition with all parts of the same size?

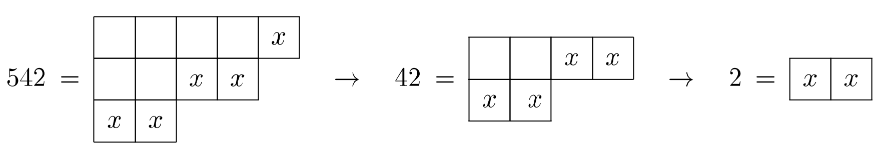
Exercise: What does the width of a Young diagram represent? What does the height represent?

If we compare Young diagrams with the composition representations above, the multiplicity of a part corresponds to the number of rows of that length, and the successive differences correspond to the lengths by which the rows overhang the rows immediately beneath them. The x's in the examples below indicate overhanging squares. The composition has a part of size 0 if a row has no x. The sum of the differences is the size of the largest part.



Exercise: Draw the Young diagrams for the partitions of 5 and smaller, find each successive difference of the parts, and verify that the segment lengths sum to the length of the top row.

An important extension of the successive difference formulation is to form a new partition from a given partition by removing the squares that would contain x's in the manner shown above. (That is, the boxes in the Young diagram which are viewable from below are removed.) By repeating this process, we get a sequence of partitions and weak compositions. For instance, 542 yields the sequence



with compositions 122, 22, and 2. Do you see how to reverse the process from the compositions? Note that the number of parts decreases by one at each step. The corresponding sequences for 7422 are partitions 7422, 422, 22, 2 and weak compositions 3202, 202, 02, 2.

Exercise: Repeat for the partitions 644 and 6542. Try more examples, and record each sequence of partitions in the shape of an equilateral triangle like the one shown at right.

$$\begin{array}{ccccccc} 7 & & 4 & & 2 & & 2 \\ & 4 & & 2 & & 2 & \\ & & 2 & & 2 & & \\ & & & 2 & & & \end{array}$$

Exercise: Construct the sequences of partitions and Young diagrams associated to the weak compositions (of successive differences) 123 and 20301. What happens if all parts of the composition are equal to 1?

Such a decomposition of a partition is a special case of a **Gelfand-Tsetlin pattern**. Gelfand-Tsetlin patterns are number arrangements in the shape of an inverted equilateral triangle such that each lower entry is numerically in between the two entries above or equal to one of them. These patterns arise in calculations in combinatorics, particle physics, and representation theory.

Exercise: Consider the sequence of partitions 6532, 542, 43, 3. Draw the Young diagram for 6532, and, in the order that squares are added (by overlaying Young diagrams for the partitions in the sequence in reverse so that their top and left edges are aligned), label the added squares at each step with 1 through 4. What Young diagrams result from restricting to squares with values less than 1, 2, 3, or 4, respectively?

Exercise: Explore Gelfand-Tsetlin patterns as follows: choose a partition, extend to a Gelfand-Tsetlin pattern, label squares as in the previous exercise, and then confirm the restriction property. What properties must a sequence of Young diagrams possess to represent a Gelfand-Tsetlin pattern?

Many other properties of partitions follow from Young diagrams. For a given Young diagram, another partition occurs if we instead list the sizes of the columns. From the above diagrams for the partitions 542 and 7422, the column lists are 33221 and 4422111, respectively. We call this operation the **conjugation** of a partition. Note that conjugation preserves the number of squares.

Exercise: Draw the conjugated partitions above and describe the conjugate of a partition geometrically. Find the conjugates of all the partitions of 5. What happens if we conjugate the conjugate of a partition?

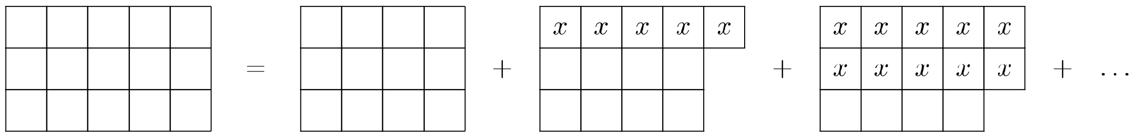
Exercise: What types of partitions are unchanged under the conjugation operation? Draw several examples. Explain how the Young diagrams for such partitions of k are in one-to-one correspondence with partitions of k with distinct odd parts. Find the generating function that counts such partitions.

Exercise: Verify that the first entry of a partition is the number of parts in the conjugate.

In the previous installment, we noted that $p_{[n]}(k)$ is also equal to the number of partitions of k with at most n parts. While we first showed this using generating functions, it now follows directly from conjugation.

Further results from the previous installment can be visualized with Young diagrams. Since partitions with largest part at most n correspond to those Young diagrams with width less than or equal to n, we interpret the hockey stick rule as follows with Young diagrams:

$$p_{[n]}(k) = p_{[n-1]}(k) + p_{[n-1]}(k-n) + p_{[n-1]}(k-2n) + \ldots.$$

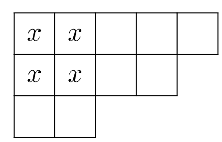


This schematic illustrates the identity at the top of the page when $n = 5$; the general case follows from a similar schematic. The first rectangle represents all Young diagrams of width at most 5 and with k squares. The other three diagrams also represent collections of Young diagrams. Each subset is determined by the number of parts of size 5 that it contains; we mark these parts with x symbols. The remaining rows allow 4 or fewer squares.

Exercise: Revisit the previous installment and interpret the other results about the partition triangle in terms of Young diagrams.

The idea behind the Young diagram proof of the hockey stick rule extends to other shapes. For the hockey stick rule, we append Young diagrams of smaller width to a progression of rectangles of width n. To remove the width condition, we instead consider a progression of squares with increasing size.

Young diagrams contain many sub-rectangles, but there is a unique square of largest size, called the **Durfee square**, that fits inside a Young diagram from the upper left-hand corner. For the example of 542 shown at right, the Durfee square has size 2.



Exercise: Find the size of the Durfee squares of the Young diagrams corresponding to the partitions 111, 54321, and 5555.

Note that every Young diagram decomposes into its Durfee square of size m, a Young diagram of height less than or equal to m, and a Young diagram of width less than or equal to m. Using the matching rule, the number of Young diagrams with k squares and a Durfee square of size m is the sum of

$$p_{[m]}(k - m^2 - s)p_{[m]}(s)$$

as s ranges from 0 to $k - m^2$. Since convolution corresponds to a product of generating functions, this count is the coefficient of t^k in

$$\frac{t^{m^2}}{(1-t)\cdots(1-t^m)}\frac{1}{(1-t)\cdots(1-t^m)} = \frac{t^{m^2}}{(1-t)^2\cdots(1-t^m)^2}.$$

If we sum over all m, we obtain the equality of generating functions

$$\frac{1}{(1-t)(1-t^2)(1-t^3)\cdots} = 1 + \frac{t}{(1-t)^2} + \frac{t^4}{(1-t)^2(1-t^2)^2} + \ldots.$$

Exercise: Expand the three terms on the right-hand side to obtain the terms up to t^8 of the generating function for partition numbers. For the second term, use the binomial series

$$\frac{1}{(1-t)^2} = 1 + 2t + 3t^2 + \ldots.$$

Then rewrite the third term as

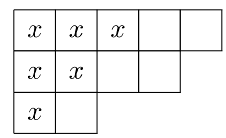
$$\frac{t^4(1+t)^2}{(1-t^2)^4}$$

and apply the binomial series

$$\frac{1}{(1-t)^4} = 1 + 4t + 10t^2 + \ldots.$$

Finally, list all partitions for k up to 6 to verify the Durfee square counts from each term.

This approach also works for partitions with distinct parts. In addition to a distinguished square, each Young diagram with distinct row lengths contains a distinguished right triangle. This triangle has sides in the first row and column of the Young diagram, and the hypotenuse is the largest diagonal to the upper right that completes such a triangle.



Exercise: How many squares are in such a triangle of height m? When does the triangle contain the Durfee square, and vice versa?

Exercise: Prove that every Young diagram with distinct row lengths decomposes uniquely into its distinguished triangle of height m and, after shifting square to the left to left justify, a Young diagram of height less than or equal to m. Also prove that a Young diagram constructed in this manner has distinct row lengths.

For Young diagrams with distinct rows, the numbers with k squares and distinguished triangle of height m is the coefficient of t^k in

$$\frac{t^{\frac{m(m+1)}{2}}}{(1-t)\cdots(1-t^m)}.$$

Since the counting functions for partition numbers with odd parts and distinct parts coincide, we obtain the equality of generating functions

$$\frac{1}{(1-t)(1-t^3)(1-t^5)\cdots} = 1 + \frac{t}{(1-t)} + \frac{t^3}{(1-t)(1-t^2)} + \frac{t^6}{(1-t)(1-t^2)(1-t^3)} + \ldots.$$

Recall that the sequence corresponding to the left-hand side is sequence A00009 in the On-Line Encyclopedia of Integer Sequences.

Exercise: Expand the four terms on the right-hand side of this equation to obtain the terms up to t^9 of the generating function for partition numbers with odd parts. List all such partitions for k up to eight and verify the counts for each distinguished triangle size.

Denote by $P_{m,n}$ the number of Young diagrams that fit into a rectangle with height m and width n. Denote by $p_{m,n}(k)$ the number of such diagrams with k squares. Of course, $P_{m,n} = P_{n,m}$ and $p_{m,n}(k) = p_{n,m}(k)$.

Exercise: Compute $P_{m,n}$ directly for rectangles with heights 1, 2, and 3.

Exercise: Give a recurrence relation for $P_{m,n}$ based on the width n. Draw the corresponding pictures with Young diagrams.

For such a Young diagram with k squares, we assign the composition of n with $m + 1$ parts whose entries are the multiplicities of row lengths 0 through m. These words are counted by the binomial coefficient

$$P_{m,n} = \binom{m+n}{m}.$$

Exercise: Give another proof of this formula by counting the paths in the rectangle traced by the lower edges of all Young diagrams that it contains.

Exercise: Prove that the formula for $P_{m,n}$ satisfies the recurrence relation.

Now consider $p_{m,n}(k)$. If we alter the argument for the hockey stick rule, we obtain the recurrence relation

$$p_{m,n}(k) = p_{m,n-1}(k) + p_{m-1,n}(k-n).$$

Let's calculate the associated generating function $F_{m,n}(t)$. First, note that $p_{1,n}(k) = 1$ if $0 \le k \le n$ and 0 otherwise. Thus

$$F_{1,n}(t) = p_{1,n}(0) + p_{1,n}(1)t + \ldots + p_{1,n}(n)t^n = 1 + t + t^2 + \ldots + t^n.$$

In terms of generating functions, the recurrence becomes

$$F_{m,n}(t) = F_{m,n-1}(t) + t^n F_{m-1,n}(t).$$

The factor of t^n acts as a shift operation. For instance, if we list only the coefficients for $F_{m,n}(t)$, so that $F_{1,2}(t)$ is given as 1 1 1, then we obtain $F_{2,2}(t)$, $F_{2,3}(t)$, and $F_{2,4}(t)$ as follows:

$$\begin{array}{ccccc} 1&1&1&&\\ &&1&1&1\\ \hline 1&1&2&1&1 \end{array} \qquad \begin{array}{ccccccc} 1&1&2&1&1&&\\ &&&1&1&1&1\\ \hline 1&1&2&2&2&1&1 \end{array} \qquad \begin{array}{ccccccccc} 1&1&2&2&2&1&1&&\\ &&&&1&1&1&1&1\\ \hline 1&1&2&2&3&2&2&1&1 \end{array}$$

Exercise: Find $F_{2,5}(t)$ and $F_{2,6}(t)$. Based on these examples, guess the general pattern, which might be familiar, and prove the general pattern for $F_{2,n}(t)$. Verify the coefficients for $F_{2,n}(t)$ with $n = 2, 3, 4$ by listing all partitions in the corresponding rectangle.

Exercise: Find the formula for $F_{3,3}(t)$ and $F_{3,4}(t)$. Verify by listing the partitions.

Exercise: Using Young diagrams, explain why the coefficients of $F_{m,n}(t)$ are symmetric. That is, consider the squares not used by a Young diagram in the rectangle.

We find the general pattern for $F_{2,n}(t)$ as a product of functions. From the recurrence,

$$F_{2,n}(t) = F_{2,n-1}(t) + t^n F_{1,n}(t) = F_{2,n-1}(t) + t^n(1 + t + t^2 + \dots + t^n).$$

With $F_{2,1}(t) = 1 + t + t^2$, we iterate to find

$$F_{2,2}(t) = (1 + t^2)(1 + t + t^2),$$
$$F_{2,3}(t) = (1 + t^2)(1 + t + t^2 + t^3 + t^4).$$

The general formulas are given by

$$F_{2,n}(t) = \begin{cases} (1 + t^2 + \dots + t^n)(1 + t + t^2 + \dots + t^n) & (n \text{ even}), \\ (1 + t^2 + \dots + t^{n-1})(1 + t + t^2 + \dots + t^{n+1}) & (n \text{ odd}). \end{cases}$$

Exercise: Verify the formulas for $F_{2,2}(t)$ and $F_{2,3}(t)$. Then, assuming the cases with $n - 1$ are true, prove the general formulas for $F_{2,n}(t)$ are true. Can you prove the formulas by describing the associated convolutions?

The general formulas can be further expanded using geometric sums. For instance, when n is even,

$$F_{2,n}(t) = \frac{(1-t^{n+1})(1-t^{n+2})}{(1-t)(1-t^2)} = \frac{(1-t^{n+1})(1-t^{n+2})(1+t)}{(1-t^2)^2}$$

and, if we convert to binomial series, we obtain

$$p_{2,n}(2k) = \binom{k+1}{1} - 2\binom{k-n/2-1}{1} + \binom{k-n-2}{1},$$
$$p_{2,n}(2k+1) = \binom{k+1}{1} - \binom{k-n/2}{1} - \binom{k-n/2-1}{1} + \binom{k-n-2}{1}.$$

We express the formulas in this way since binomial coefficients vanish when the upper index is negative or smaller than the lower index.

Exercise: Verify these formulas for all values of k when $n = 2, 4$. For general even n, show vanishing when $k > n$. With n fixed, draw the graphs of $p_{2,n}$ as functions of k.

Exercise: Find the analogous formulas for odd n, and repeat the previous exercise with $n = 1, 3$.

Follow Your Nose

by Ken Fan | edited by Jennifer Sidney

As Prof. Lange[2] experienced, with math you don't always have to read it in a book. *You* can figure it out.

Last month, I followed a group of 8th graders who took it upon themselves to find all of the Pythagorean triples. Would they succeed?

A Pythagorean triple consists of three whole numbers that correspond to the lengths of the sides of a right triangle. The most famous Pythagorean triple is 3, 4, and 5. But there are many others, such as 5, 12, and 13, or 6, 8, and 10. These numbers satisfy the famous Pythagorean equation that relates the side lengths of a right triangle with hypotenuse of length c and legs of lengths a and b:

$$a^2 + b^2 = c^2.$$

Any positive numbers a, b, and c that satisfy this equation correspond to the lengths of the sides of a right triangle, with c being the length of the hypotenuse.

The problem of finding all of the Pythagorean triples is well known and was worked out long ago. I just googled "Pythagorean triple" and got back 2,110,000 hits!

But you don't have to look them up. You can figure them out, and you might discover how fun it is to do so. Try it!

If you're skeptical, let's look at what these 8th graders did. As we follow their work, ask yourself, "Is there anything they did that I couldn't do myself?"

Examples

The first thing they decided to do was to find more examples of Pythagorean triples. With more examples, they reasoned, they might detect a pattern.

Using a combination of recall and guessing, they added (8, 15, 17), (7, 24, 25), and (9, 12, 15) to the solutions (3, 4, 5), (5, 12, 13), and (6, 8, 10).

An Observation

One of them noticed that in three of the solutions, the two larger numbers differed by 1: (3, 4, 5), (5, 12, 13), and (7, 24, 25). They wondered if there were other Pythagorean triples a, b, and c where $c - b$ was equal to 1.

To find out, they substituted $b + 1$ for c in the Pythagorean equation to get

$$a^2 + b^2 = (b + 1)^2.$$

[2] See our interview with Professor Karen Lange on page 3.

By expanding $(b + 1)^2$, simplifying, and rearranging terms, they rewrote this equation as $a^2 - 1 = 2b$. From this equation, they saw that if $a^2 - 1$ is even, then b would be an integer. And they saw that $a^2 - 1$ is even if and only if a is odd.

They plugged in consecutive odd integers for a to get more Pythagorean triples. When $a = 1$, they got the degenerate solution (1, 0, 1); but for bigger odd numbers, they rediscovered (3, 4, 5), (5, 12, 13), and (7, 24, 25), then found the solutions (9, 40, 41), (11, 60, 61), (13, 84, 85), etc.

Isn't that neat? In just a few minutes, they managed to find an infinite family of Pythagorean triples all by themselves! Is there anything they did that you don't think you could have done?

A Sensible Next Step

Emboldened by this win, they decided to try to find solutions where $c - b = 2$.

Guided by what they had just done, they substituted $b + 2$ for c in the Pythagorean equation and simplified to find the equation

$$a^2 - 4 = 4b.$$

This equation made them wonder, "For which values of a is $a^2 - 4$ divisible by 4?"

They reasoned that for $a^2 - 4$ to be divisible by 4, a^2 must be divisible by 4, and a^2 is divisible by 4 whenever a is even.

By substituting consecutive even numbers for a starting at 2, they found the degenerate solution (2, 0, 2), recovered the triples (4, 3, 5), (6, 8, 10), and (8, 15, 17), and found (10, 24, 26), (12, 35, 37), (14, 48, 50), etc.

What do you think they did next?

If you guessed that they tried to look for solutions where $c - b = 3$, you're right.

But they stopped midway through their procedure, because one of them had this wonderful thought: Instead of doing $c - b = 3$, then $c - b = 4$, $c - b = 5$, and so on, what if we try to do all of the cases by doing $c - b = x$? If we can do that, then we can substitute 1, 2, 3, … for x to get all of these special cases.

In other words, they utilized the concept of the variable!

So instead of substituting $b + 1$, $b + 2$, or $b + 3$ for c, they substituted $b + x$ in the Pythagorean equation and found this intriguing equation:

$$a^2 - x^2 = 2xb.$$

A few of them recalled the "difference of squares" algebraic identity and applied it to rewrite this equation as

$$(a + x)(a - x) = 2xb.$$

So, now the question became: For what values of a and x is $(a + x)(a - x)$ divisible by $2x$?

Isn't it neat that a question about right triangles morphed into a specific question about divisibility?

Unlike for the cases $x = 1$ and 2, the solution to the general divisibility problem was not clear. They decided to make a table, with rows corresponding to different values of x; in each row, they systematically listed values of a for which $2x$ divides evenly into $(a + x)(a - x)$.

x	First several values of a for which $(a + x)(a - x)$ is divisible by $2x$
1	1 3 5 7 9 11 13 15 17 19 21 23 25 27 29 31 33 35 37 39
2	2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40
3	3 9 15 21 27 33 39 45 51 57 63 69 75 81 87 93 99 105 111 117
4	4 8 12 16 20 24 28 32 36 40 44 48 52 56 60 64 68 72 76 80
5	5 15 25 35 45 55 65 75 85 95 105 115 125 135 145 155 165 175 185 195
6	6 12 18 24 30 36 42 48 54 60 66 72 78 84 90 96 102 108 114 120
7	7 21 35 49 63 77 91 105 119 133 147 161 175 189 203 217 231 245 259 273
8	4 8 12 16 20 24 28 32 36 40 44 48 52 56 60 64 68 72 76 80
9	3 9 15 21 27 33 39 45 51 57 63 69 75 81 87 93 99 105 111 117
10	10 20 30 40 50 60 70 80 90 100 110 120 130 140 150 160 170 180 190 200

What patterns do you detect?

The 8th graders noticed that sometimes, the values of a that work are the odd multiples of x, such as when $x = 1$, 3, 5, and 7. But in other instances, there are values of a that work that are not multiples of x, such as when $x = 8$ or 9. Yet even when the values of a that worked were not multiples of x, they were still spaced evenly, forming arithmetic series.

Then, one of the 8th graders noticed that the product $(a + x)(a - x)$ is a product of two numbers that differ by $2x$, the very same number that the product must be divided by to obtain b. That is, upon division by $2x$, $a + x$ and $a - x$ leave the same remainder. Let's call this remainder r. This means that $a - x$ is r more than some multiple of $2x$. In other words, there is an integer m such that $a - x = 2xm + r$, which implies $a + x = 2x(m + 1) + r$. Using these expressions for $a - x$ and $a + x$, the students needed to determine when $(2x(m + 1) + r)(2xm + r)$ is divisible by $2x$. Now,

$$(2x(m + 1) + r)(2xm + r) = 4x^2m(m + 1) + 2xr(m + 1) + 2xrm + r^2.$$

Given that the first three terms of the right side of this equation are multiples of $2x$, it suffices to determine values of r such that r^2 is divisible by $2x$.

Other students were thinking about prime factorizations, since the question of whether a number Y is divisible by a number X becomes clear if we know the prime factorizations of Y and X. We need only check that for each prime number p, the exponent of p in the prime factorization of X is less than or equal to its exponent in the prime factorization of Y.

So let $2^{n_1} p_2^{n_2} p_3^{n_3} p_4^{n_4} \cdots p_k^{n_k}$ be the prime factorization of $2x$. Here, we have made the prime factor of 2 explicit because 2 must appear with positive exponent in the prime factorization of $2x$. And let $2^{m_1} p_2^{m_2} p_3^{m_3} p_4^{m_4} \cdots p_k^{m_k}$ be the prime factorization of r. Again, we make the prime factor of 2 explicit and use the same list of primes p_2 through p_k for both by allowing the possibility of having exponents that are zero. Then the prime factorization of r^2 is $2^{2m_1} p_2^{2m_2} p_3^{2m_3} p_4^{2m_4} \cdots p_k^{2m_k}$.

In order for r^2 to be divisible by $2x$, we must have $2m_i \geq n_i$ for $i = 1, 2, 3, \ldots, k$. In other words, if we define

$$r_0 = 2^{\lceil n_1/2 \rceil} p_2^{\lceil n_2/2 \rceil} p_3^{\lceil n_3/2 \rceil} p_4^{\lceil n_4/2 \rceil} \cdots p_k^{\lceil n_k/2 \rceil},$$

where $\lceil x \rceil$ is the least integer greater than or equal to x (also known as the **ceiling** of x), then r must be divisible by r_0; conversely, if r is a multiple of r_0, then r^2 will be divisible by $2x$.

Success!

In this way, the 8th graders found that for any positive integer m, we can let $a - x = mr_0$ and get a Pythagorean triple! Specifically, we get the solution

$$a = mr_0 + x, \quad b = mr_0(mr_0 + 2x)/(2x), \quad c = b + x.$$

Let's apply this to the case $x = 3$, which they were about to do before they shifted their focus to the general method. When $x = 3$, $2x = 6 = 2 \cdot 3$, so $r_0 = 2 \cdot 3$. Thus, values of a for which $(a + x)(a - x)$ is divisible by $2x$ are $a = 6m + 3$, for any integer m (consistent with the table). For this value of a, we find $b = 6m(6m + 6)/6 = 6m(m + 1)$, and $c = 6m(m + 1) + 3$. For $m = 1, 2, 3$, etc., we find the Pythagorean triples (9, 12, 15), (15, 36, 39), (21, 72, 75), etc.

When $x = 4$, $2x = 8 = 2^3$, so $r_0 = 2^2 = 4$. Therefore, $a = 4m + 4$, $b = 4m(4m + 8)/8 = 2m(m + 2)$, and $c = 2m(m + 2) + 4$. We find the Pythagorean triples (8, 6, 10), (12, 16, 20), (16, 30, 34), (20, 48, 52), (24, 70, 74), (28, 96, 100), (32, 126, 130), etc.

The 8th graders succeeded in devising an algorithm for producing every single Pythagorean triple! Their method does not use trial and error. And they did it without having to look in a book!

Is there anything they did that you think you couldn't do yourself? I'd bet not!

With a little determination, I am sure you would be able to come up with your own method for producing Pythagorean triples. One key to succeeding is not to dismiss your own thoughts. When you have an idea, try it! Notice that the initial ideas these 8th graders had did not immediately lead to a general solution; they got to the general solution bit by bit. So when you attempt your idea, don't expect to find a complete solution initially. Even if you don't get a complete solution, your efforts will very likely give you other ideas.

Primitivity

The students noticed that many of the solutions they were getting were scaled up from smaller solutions. For example, (6, 8, 10) is scaled up from (3, 4, 5) by a factor of 2. Also, (3, 4, 5) and (4, 3, 5) are essentially the same solution, but the algorithm would produce the first if you set x to 1 and the second if you set x to 2. So their algorithm leads to further questions: For which values of x and m will the resulting solution be **primitive**, that is, will not be a multiple of a smaller solution? For which values of x and m will we have $a < b < c$? (Can you show that in any Pythagorean triple with $a^2 + b^2 = c^2$, we cannot have $a = b$?)

At this point, however, the students voted to switch gears and pursue other questions.

I'll pick up their investigation where they left off to show how it could have unfolded. But if you're interested in seeing how you might expand upon their work, read no further and have fun!

Primitive Pythagorean Triples

Let's try to determine which values of x and m produce primitive Pythagorean triples. In the equation $a^2 + b^2 = c^2$, notice that any prime number that divides evenly into any two of the numbers a, b, or c must also divide the third. For example, if a prime number p divides both a and b, then p divides a^2 and b^2; hence p divides $a^2 + b^2$, which is c^2. And if a prime number divides c^2, then it must divide c. (Please check the other two cases!) This means that in a primitive Pythagorean triple, the numbers are pairwise relatively prime, and if any two numbers are relatively prime, then the triple is primitive.

So to understand which values of x and m produce a primitive Pythagorean triple, it suffices to determine when any two of a, b, or c are relatively prime.

Let's recall the formulas the students found for a, b, and c:

$$a = mr_0 + x \qquad b = mr_0(mr_0 + 2x)/(2x) \qquad c = b + x$$

Suppose we wish to understand whether two numbers X and Y are relatively prime. If X and Y have a common divisor greater than 1, they will have a common prime divisor. But if X and Y are relatively prime, then no prime will divide evenly into both. Thus, to check for relative prime-ness, we can proceed prime number by prime number. That is, for each prime number p, we can check to see if p divides both X and Y.

We might as well start with the first prime number, 2.

If x is odd, then – in the notation from earlier – we have $n_1 = 1$ (recall that n_1 is the exponent of 2 in the prime factorization of $2x$). Here r_0 will have a factor of 2, but not 4. This means mr_0 is even, and since we're assuming that x is odd, it must be that a is odd. Because $c = b + x$, one of b and c will be even and the other will be odd. Therefore, 2 will not be a common factor of a, b, and c.

If x is even, then $n_1 > 1$ and r_0 will be even. That means a is even. Since b and c now have the same parity, we need only determine if b is even or not. We can rewrite the expression for b as

$$b = m^2\frac{r_0^2}{2x} + mr_0. \tag{*}$$

By construction, $r_0^2/(2x)$ is a whole number. Since m is a factor in both terms, if m is even, then b will be even and the triple will not be primitive. So let's assume m is odd. We know r_0 is even, so b will be odd only if $r_0^2/(2x)$ is odd, and $r_0^2/(2x)$ is odd if and only if n_1 is even.

Putting this together, we've found that the triple a, b, c will not have a common factor of 2 if and only if x is odd or if 2 appears in the prime factorization of x an odd number of times and m is odd. Otherwise, all three numbers a, b, and c will be even.

Now suppose p is an odd prime number.

Suppose p does not divide evenly into x. If p divided evenly into both b and c, then p would divide their difference; but $c - b = x$, and that would be a contradiction.

So suppose that p does divide into x. Let n be the exponent of p in the prime factorization of x. Since $n > 0$, p divides r_0 and, hence, p divides a. Because b and c differ by x, which is a multiple of p, they will either both be divisible by p or neither will be divisible by p. To find a primitive triple, we would need neither to be divisible by p. We'll consider two cases: n is odd, or n is even.

If n is odd, then p^{n+1} will divide r_0^2, so p will divide $r_0^2/(2x)$. Since p also divides r_0, b will be divisible by p, and the triple will not be primitive.

If n is even (and not zero), then p does not divide $r_0^2/(2x)$. Since p does divide r_0, from formula (*), we conclude that p divides b if and only if p divides m.

Thus a, b, and c will not have a common odd prime factor p if and only if p appears with even exponent in the prime factorization of x and m is not divisible by p.

Putting all of this information about individual prime numbers together, we find that the values x and m give rise to a primitive Pythagorean triple a, b, c if and only if

- x is an odd perfect square and m is relatively prime to x
- x is an odd power of 2 times an odd perfect square and m is relatively prime to x

Note that a number that is an odd power of 2 times an odd perfect square is twice a perfect square. So we can restate the condition as follows:

> The values x and m give rise to a primitive Pythagorean triple a, b, c if and only if x is either an odd perfect square or twice a perfect square, and m is relatively prime to x.

To illustrate, let's take $x = 50 = 2\cdot5^2$ and $m = 17$. Then $r_0 = 10$ and $a = 220$, $b = 459$, and $c = 509$, and, indeed, $220^2 + 459^2 = 509^2$.

A Corollary

A corollary of this investigation is that in any primitive Pythagorean triple, the difference between the length of a leg and the length of the hypotenuse will always be either an odd perfect square or twice a perfect square.

Using the material in this investigation, can you show that in any primitive Pythagorean triple a, b, c with $a^2 + b^2 = c^2$, exactly one of a or b must be even and c must be odd? Combining this with the corollary, we conclude that the difference between the lengths of the hypotenuse and one leg will be an odd perfect square, while the difference between the lengths of the hypotenuse and the other leg will be twice a perfect square. For example, in 220, 459, 509, the difference $509 - 220$ is 289, which is 17^2, whereas $509 - 459 = 2\cdot5^2$. Isn't that neat?

Romping Through the Rationals, Part 5

by Ken Fan | edited by Jennifer Sidney

Emily and Jasmine are studying sequences a_n of nonnegative integers that have the property that consecutive terms are relatively prime and every nonnegative rational number is equal to a_n/a_{n+1} for a unique n. They have dubbed these sequences "rational rompers."

Last time, they seized upon an idea that they are hoping will enable them to transform any rational romper into any other rational romper by a sequence of operations that they call a "splice." A splice modifies a rational romper a_n in the following way: Suppose x, y are consecutive terms in the sequence a_n and suppose there is a subsequence disjoint from the consecutive terms x, y, but which also begins with x and ends with y. That is, suppose $a_k = x$ and $a_{k+1} = y$, and there is a subsequence $a_p, a_{p+1}, a_{p+2}, \ldots, a_q$, where we have either $p > k + 1$ or $q < k$. Then we can obtain another rational romper by removing the subsequence $a_{p+1}, \ldots, a_{q-1}$ and reinserting it between a_k and a_{k+1}.

Jasmine: We've managed to show that we can use the splicing operation to modify any rational romper so that it begins 0, 1, p, where p is a positive integer. Now I think we should try to show that any rational romper can be transformed into any other rational romper by a sequence of splices inductively, by showing that we can modify one to agree with the other on its first 3 terms, then its first 4 terms, then its first 5 terms, etc.

Emily: Sounds like a good strategy! So suppose a_n and b_n are two rational rompers. Since all rational rompers must begin 0, 1, …, we know that a_n and b_n have the same first two terms. So now let's assume that $a_k = b_k$ for all $k < N$, where N is some positive integer greater than 2, and suppose $a_N \neq b_N$. Can we show that we can modify the sequence a_n using the splicing operation so that the resulting sequence agrees with the sequence b_n on its first N terms?

Jasmine: Let's see. In that setup, the two sequences begin like this:

$$a_1, a_2, a_3, \ldots, a_{N-1}, a_N, a_{N+1}, \ldots$$

and

$$a_1, a_2, a_3, \ldots, a_{N-1}, b_N, b_{N+1}, \ldots.$$

For definiteness, let's also assume that $a_{N-1} \neq a_N$. If that's not the case, we can just swap the labels of the two sequences.

Emily: Okay. We don't want to mess with the first $N - 1$ terms anymore, so hopefully we can do some kind of splice that changes that a_N to b_N and only involves moving around parts of the sequence beyond the first $N - 1$ terms.

Jasmine: If we're lucky enough to be able to perform a single splicing operation to make the desired change, we would need to find, in the first sequence, the consecutive terms a_{N-1}, b_N somewhere after the first N terms.

Emily: And that will happen!

Jasmine: Why can't the subsequence a_{N-1}, b_N occur among the first N terms?

Emily: Because the two sequences agree on the first $N - 1$ terms, and in the second sequence, a_{N-1}, b_N occurs as the $(N - 1)$-th and Nth terms. Since it's a rational romper, that must be the one and only time those two numbers appear consecutively. So a_{N-1}, b_N does not occur among the first $N - 1$ terms of either sequence!

Jasmine: Oh, nice! So let's say that $a_m = a_{N-1}$ and $a_{m+1} = b_N$, with $m \geq N$. In fact, we can assume that $m > N$, because we've set up our sequences so that $a_{N-1} \neq a_N$. So our first sequence goes like this: $a_1, a_2, a_3, \ldots, a_{N-1}, a_N, \ldots, a_m = a_{N-1}, a_{m+1} = b_N, a_{m+2}, \ldots$.

Emily: We'd like to perform a splice that moves the $(m + 1)$-th term to the Nth term.

Jasmine: That means we want to find the first occurrence of a_N after the $(m + 1)$-th term. And since any positive integer appears infinitely many times in a rational romper, there will be an M which is the smallest positive integer greater than $m + 1$ such that $a_M = a_N$:

$$a_1, a_2, a_3, \ldots, a_{N-1}, a_N, \ldots, a_m = a_{N-1}, a_{m+1} = b_N, \ldots, a_M = a_N, a_{M+1}, \ldots.$$

I think we're all set to perform the splice!

Emily: Yes, we can remove the subsequence of terms between the mth term and the Mth term and reinsert it between the $(N - 1)$-th term and the Nth term:

$$a_1, a_2, a_3, \ldots, a_{N-1}, a_{m+1} = b_N, \ldots, a_{M-1}, a_N, \ldots, a_m = a_{N-1}, a_M = a_N, a_{M+1}, \ldots.$$

The result is a sequence that agrees with the rational romper b_n on its first N terms! Since we can always extend the terms on which the sequences agree, by induction, any rational romper can be transformed into any other rational romper by a sequence of splices!

Jasmine: Wow, that was painless! There was no obstruction.

Emily: I'm kind of stunned how everything just works out.

Jasmine: But it may take an infinite number of splice operations to effect the transformation; I guess that's to be expected since two rational rompers might differ in infinitely many places.

Emily: Actually, when can a rational romper be transformed to another rational romper via a finite sequence of splices?

Jasmine: Hmm. If we perform a finite number of splices, there would have to be a point in the sequence beyond which none of the terms are affected by any of the splices, because each splice affects the positions of only finitely many terms. So if a_n and b_n are two rational rompers that can be transformed into each other via a finite sequence of splices, they would have to agree on an infinite tail of terms.

Emily: It seems like the converse might be true, too… that two rational rompers that agree on an infinite tail of terms must be related by a finite sequence of splices. But it's not clear to me that the method we used to transform one rational romper to another won't keep messing up terms and require an infinite number of splices in all cases.

Jasmine: Maybe we can prove it by induction in the other direction. Suppose we have two rational rompers a_n and b_n and $a_k = b_k$ for all $k > N$ for some positive integer N. If we can show that we can perform a splice to the sequence a_n so that the result agrees with b_n on all terms from the Nth one on, that would do it.

Emily: To do that, we would probably want to perform splices that move around only terms before the $(N + 1)$-th term. If that's possible, then I think it means that splicing should enable us to transform any *finite* "rational romper" to any other finite rational romper that represents the same set of rational numbers.

Jasmine: I see what you're saying. You're saying that if a_n and b_n are finite sequences of the same length and consecutive terms are relatively prime, and the collection of rational numbers a_k/a_{k+1} is distinct and forms the same set of rational numbers as the rational numbers b_k/b_{k+1}, then you want to be able to transform a_n to b_n via a finite sequence of splices?

Emily: Yes, although I think we also want to require that the two sequences have the same last term, since no splice can move the last term of a finite sequence to another location in the sequence. This additional constraint isn't a concern because with actual rational romper sequences that agree on an infinite tail of terms, we can apply it to the first so many terms up to and including the first term of the tails where they agree.

Jasmine: Maybe the same inductive argument we already came up with will work for these finite sequences. Let's see.

Emily and Jasmine review their inductive argument.

Emily: I can see that the argument goes through up to where we show that the first sequence looks like this: $a_1, a_2, a_3, \ldots, a_{N-1}, a_N, \ldots, a_m = a_{N-1}, a_{m+1} = b_N, a_{m+2}, \ldots$. But then we have to find an occurrence of a_N after the $(m + 1)$-th term, and I don't see why that has to be true in the finite case. In the infinite case, there's no problem because every positive integer must appear infinitely many times in the sequence.

Jasmine: Actually, I think there's a counterexample. Consider the sequence 0, 1, 2, 1, 3, 1, 4 and the sequence 0, 1, 3, 1, 2, 1, 4. Both represent the rational numbers 0, 1/2, 2, 1/3, 3, and 1/4, but there's no way to perform splices to turn one into the other. In fact, no splice can be performed on either sequence!

Emily: Oh dear. What can be done?

To be continued …

Notes from the Club

These notes cover some of what happened at Girls' Angle meets. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

Session 33 - Meet 1 September 14, 2023	Mentors:	Elisabeth Bullock, Jade Buckwalter, Anushree Gupta, Shauna Kwag, Bridget Li, Gautami Mudaliar, Hanna Mularczyk, AnaMaria Perez, Vievie Romanelli, Swathi Senthil, Padmasini Venkat, Jing Wang

We welcome all new and returning members and mentors to our 17th year of Girls' Angle!

Two separate groups of members happened to be pondering the same topic: How does one find the equations of tangent lines to conic sections. One group was working on parabolas while the other was working on ellipses. Members of neither group knew about calculus. There are different ways of solving this problem. Can you think of a way?

Session 33 - Meet 2 September 21, 2023	Mentors:	Elisabeth Bullock, Jade Buckwalter, Anushree Gupta, Bridget Li, Gautami Mudaliar, Hanna Mularczyk, Tharini Padmagarisan, AnaMaria Perez, Vievie Romanelli, Swathi Senthil, Padmasini Venkat, Jane Wang, Jing Wang

Some members worked on solving contest problems from the 2022 AMC 10A mathematics competition. If you enjoy solving problems from past competitions, we suggest that when you solve them, you try to

- solve the problem in more than one way.
- see how much you can solve of the problem entirely in your head.
- explicitly identify the central idea(s) of the problem.
- understand how the answer depends on the given information.
- modify the problem or create a related problem.

Session 33 - Meet 3 September 28, 2023	Mentors:	Elisabeth Bullock, Jade Buckwalter, Anushree Gupta, Gautami Mudaliar, Hanna Mularczyk, Tharini Padmagarisan, AnaMaria Perez, Vievie Romanelli, Swathi Senthil, Jing Wang, Julia Wei, Dora Woodruff
	Visitor:	Isabel Vogt, Brown University

We're thrilled to have a Support Network visit from Isabel Vogt, assistant professor of mathematics at Brown University and former Girls' Angle mentor. Isabel explained her path into mathematics and one of her most recent theorems.

She was raised in South Florida and attended a middle school that specialized in the arts. In High School, she began to get more interested in science and math. On a whim, she applied to the Summer Workshop in Mathematics at Princeton University. She got in, and enjoyed the program a lot, but when she arrived at Harvard for college, she majored in physics and chemistry. During her sophomore year, she took a math class from Prof. Joe Harris, and that's when things clicked: She realized that math was what she wanted to do. She joined the Undergraduate Women in Math club at Harvard and started mentoring at Girls' Angle.

She then attended MIT for graduate school, studying algebraic geometry under the supervision of Bjorn Poonen (who also serves on the Girls' Angle Advisory Board). There, she first encountered a problem that would interest her to this day: to understand the number of generic points on a curve of degree d, in an ambient space of dimension r, and with genus g, over the field of complex numbers. This question has its roots in the Euclidean observations that a line interpolates 2 points and a circle interpolates 3 points. That is, through any 2 points, there exists a line, but not necessarily through any 3 points, and, generically (meaning, not collinear), for any 3 points, there exists a circle through all 3, but through 4 generic points. There was a known formula for the expected number of generic points, but it was not fully proven. Over the course of years of tackling the problem and in collaboration with Eric Larson, she eventually settled the question, showing that the expected number is, in fact, correct, except for 4 exceptions with $(d, g, r) = (5, 2, 3), (6, 4, 3), (7, 2, 5)$, and $(10, 6, 5)$. *Quanta Magazine* recently wrote about her work with Larson in an article entitled "Old Problem About Mathematical Curves Falls to Young Couple."

Session 33 - Meet 4 October 5, 2023	Mentors:	Elisabeth Bullock, Jade Buckwalter, Anushree Gupta, Gautami Mudaliar, Hanna Mularczyk, Tharini Padmagarisan, AnaMaria Perez, Vievie Romanelli, Padmasini Venkat, Jane Wang, Dora Woodruff, Angelina Zhang

Suppose you have a rectangular grid that is n squares by m squares. How many ways are there to place the numbers 1 through nm into the squares in the grid in such a way that the numbers increase down any column or from left to right across any row?

Session 33 - Meet 5 October 12, 2023	Mentors:	Elisabeth Bullock, Gautami Mudaliar, Hanna Mularczyk, Tharini Padmagarisan, AnaMaria Perez, Swathi Senthil, Padmasini Venkat, Jing Wang, Dora Woodruff, Angelina Zhang

Can you make perspective drawings of all the Platonic solids? Of the five Platonic solids, the cube is probably the easiest, but even that is a challenge. How do you ensure that the drawing represents a block with equal edge lengths?

Session 33 - Meet 6 October 19, 2023	Mentors:	Jade Buckwalter, Gautami Mudaliar, Hanna Mularczyk, AnaMaria Perez, Vievie Romanelli, Swathi Senthil, Padmasini Venkat, Jing Wang, Dora Woodruff, Saba Zerefa, Angelina Zhang

How many different algorithms can you devise to sort a list of numbers?

Session 33 - Meet 7 October 26, 2023	Mentors:	Anushree Gupta, Shauna Kwag, Gautami Mudaliar, Hanna Mularczyk, Tharini Padmagarisan, AnaMaria Perez, Swathi Senthil, Padmasini Venkat, Jane Wang, Dora Woodruff, Saba Zerefa, Angelina Zhang

Can you devise a method to fold an origami regular octagon? How can you ensure that all 8 sides are the same length and that all the angles have the same measure?

Calendar

Session 33: (all dates in 2023)

September	14	Start of the thirty-third session!
	21	
	28	Support Network Visitor: Isable Vogt, Brown University
October	5	
	12	
	19	
	26	
November	2	
	9	
	16	
	23	Thanksgiving - No meet
	30	
December	7	

Session 34: (all dates in 2024)

February	1	Start of the thirty-fourth session!
	8	
	15	
	22	No meet
	29	
March	7	
	14	
	21	
	28	No meet
April	4	
	11	
	18	No meet
	25	
May	2	
	9	

Girls' Angle has run over 150 Math Collaborations. Math Collaborations are fun, fully collaborative, math events that can be adapted to a variety of group sizes and skill levels. We now have versions where all can participate remotely. We have now run four such "all-virtual" Math Collaboration. If interested, contact us at girlsangle@gmail.com. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

Girls' Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ____________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

The $50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- ☐ Enclosed is a check for $50 for a 1-year Girls' Angle Membership.
- ☐ I am making a tax-free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

A Math Club for Girls

Girls' Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is \$20/meet for members and \$30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle is based in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, founder and director of the Exploratory
Yaim Cooper, Institute for Advanced Study
Julia Elisenda Grigsby, professor of mathematics, Boston College
Kay Kirkpatrick, associate professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, assistant dean and director teaching & learning, Stanford University
Lauren McGough, postdoctoral fellow, University of Chicago
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, associate professor, University of Utah School of Medicine
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Liz Simon, graduate student, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, associate professor, University of Washington
Karen Willcox, Director, Oden Institute for Computational Engineering and Sciences, UT Austin
Lauren Williams, professor of mathematics, Harvard University

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high-level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: _______________________

(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is $216 for one session (12 meets)	☐ I will pay on a per meet basis at $30/meet.
☐ I will pay on a per meet basis at $20/meet.	☐ I'm including $50 to become a member, and I have selected an item from the left.

☐ I am making a tax-free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: ________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____________________________________