Girls' Angle Bulletin

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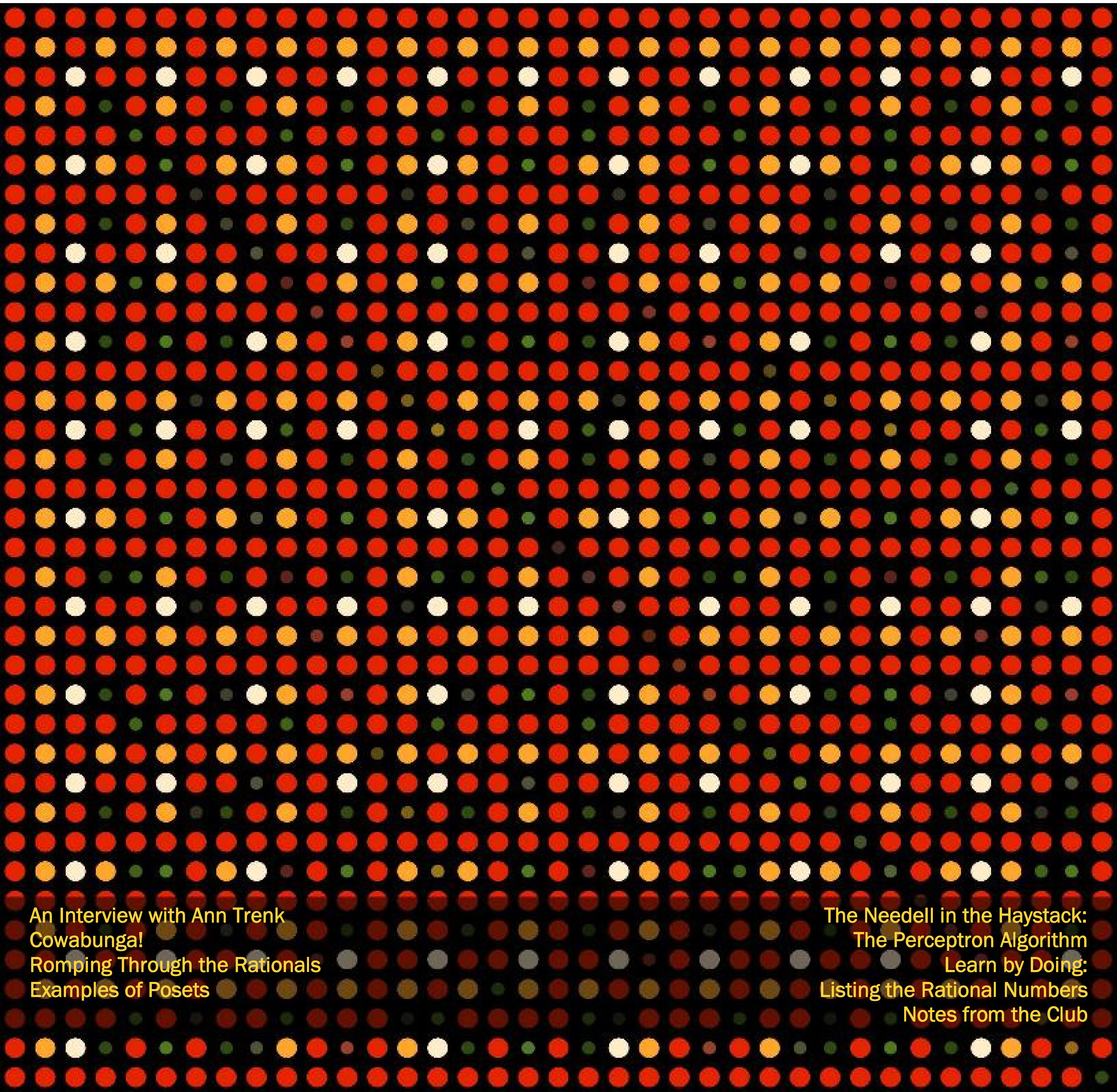
To Foster and Nurture Girls' Interest in Mathematics



From the Founder

Thank you, Google Cambridge, for hosting us. It's wonderful to be back to in-person meets, and it's tremendously inspiring to meet in such a stimulating environment. The math is flowing again after years of being stunted by the virtual environment. - Ken Fan, President and Founder

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Girls' Angle Bulletin

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Girls' Angle welcomes submissions that pertain to mathematics.

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Editors: Amanda Galtman
Jennifer Sidney
Executive Editor: C. Kenneth Fan

Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: *Greatest Common Divisors* by C. Kenneth Fan. Do you see the self-similarity? Also, see page 24.

An Interview with Ann Trenk

Ann Trenk is Lewis Atterbury Stimson Professor of Mathematics at Wellesley College. She earned her Doctor of Philosophy in Mathematics from the Johns Hopkins University under the supervision of Edward Scheinerman. She joined the faculty in the mathematics department at Wellesley College in 1992 and has served as the chair. She has taught at the Hampshire College Summer Studies in Math (HCSSiM).

This interview was conducted by Raegan Phillips and Ken Fan.

Girls' Angle: You're an expert on graph theory. What exactly is graph theory?

Ann: The "graphs" in graph theory consist of vertices (drawn as big dots), some of which are joined by edges. Graphs show relationships between objects. For example, you could form a friendship graph for a group of people. Each person would be represented by a vertex, and we would join two vertices by an edge precisely when those two people are friends. Having a visual representation of relationships can make it easier to understand what the relationships are. Another example is airline routes. The vertices are the cities the airline services, and there is an edge between two cities if there is a direct flight from one to the other. These graphs are often shown as route maps in the airline magazines you find in your seat pocket when you fly.

Girls' Angle: You co-authored a book on tolerance graphs. What are tolerance graphs?

I'm a big fan of enrichment rather than acceleration. Rather than accelerating to take calculus early in high school, I recommend broadening your exposure to math.

Ann: Before we get to tolerance graphs, I need to define interval graphs. Interval graphs are a special kind of graph that you can think of as arising from time intervals. Suppose you have 5 events you would like to attend, and each has a specific meeting time (e.g., Girls' Angle meeting, soccer game, birthday party, movie, ice cream outing). You can form a graph from these events by making a vertex for each event, and putting an edge between two events if they conflict, that is, if their time intervals overlap. If the soccer game is 2-3:45 PM and the birthday party is 3:30-5 PM, they would conflict and we would put an edge between those two vertices. Graphs that arise in this way are called interval graphs. Because they arise in this particular way, they have special properties that not all graphs have.

Tolerance graphs allow for a little more flexibility. Each event has not only a time interval, but also an amount of time for which it allows an overlap with another event. In the example above, if both the soccer game and the birthday party have a tolerance of 15 minutes or more, then they would no longer conflict. Tolerance graphs are a larger class of graphs than interval graphs, so it is interesting to see which of the special properties of interval graphs still hold for this larger class of graphs.

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We're committed to producing quality math educational content and make every effort to provide this content to you for free.

We are also committed to surviving as a nonprofit!

For this issue, those who do not subscribe to the print version will be missing out on a portion of this interview with Prof. Ann Trenk and other content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

We know that mathematical interest and talent is unrelated to economic status, which is why we provide so much content for free. But we hope that those of you who are comfortable financially will help us to continue in our efforts.

So, please consider subscribing to the Bulletin. Thanks to our sponsors, subscriptions cost $36/year. With a subscription, you have also gained access to our mentors via email and the ability to influence content in this Bulletin. Visit www.girlsangle.org/page/bulletin_sponsor.html for more information.

Thank you and best wishes,
Ken Fan
President and Founder
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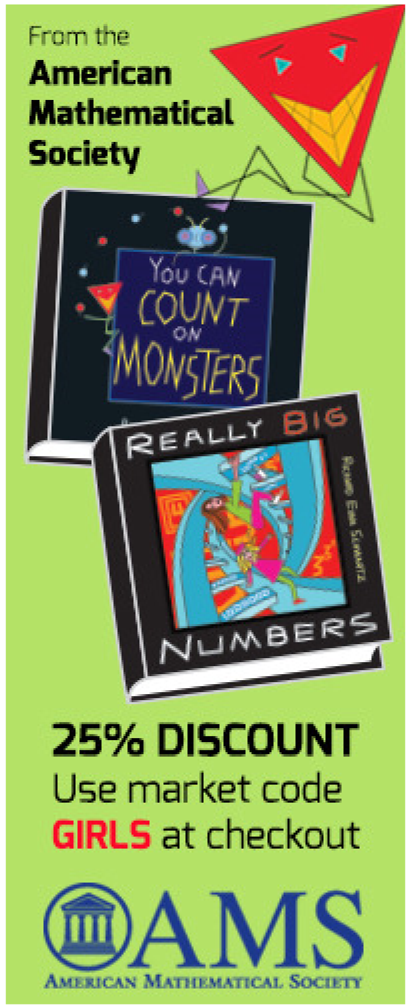

The American Mathematical Society is generously offering a 25% discount on the two book set *Really Big Numbers* and *You Can Count On Monsters* to readers of this Bulletin. To redeem, go to http://www.ams.org/bookstore-getitem/item=MBK-84-90 and use the code "GIRLS" at checkout.

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America's Greatest Math Game: Who Wants to Be a Mathematician.

(advertisement)

Cowabunga!

by Emily Caputo

Emily Caputo is a second year at Belmont High School. She created these equations for a school math assignment.

Graph these equations and inequalities for a nifty surprise! Equations and inequalities in the same row are to be satisfied simultaneously, whereas those in different rows are independent of each other. For the solution, see page 29.

$(x-6)^2+(y-6)^2=1$
$x=6,\ y=6$
$(x-4.5)^2+(y-6)^2=1,\ x<5.25,\ y>6-\sqrt{3}/2$
$x=4.5,\ y=6$
$x-3.5=2(y-7)^2,\ x<3.72$
$x=3.72,\ 22/3<y<10.75$
$y=1.5\lvert x-4.25\rvert+10,\ y<10.75$
$y=1.5\lvert x-5.25\rvert+10,\ y<10.75$
$y=1.5\lvert x-6.25\rvert+10,\ y<10.75$
$y=1.5\lvert x-7.25\rvert+10,\ y<10.75$
$y=1.5\lvert x-8.25\rvert+10,\ y<10.75$
$y-5=23(x-8.5),\ 5<y<10.75$
$(x-8.5)^2+(y-4.5)^2=0.25,\ x>8.3$
$y=5.26,\ 4.25<x<5.25$
$(x-4.25)^2+(y-5)^2+0.07,\ x<4.25$
$y=4.735,\ 4.25<x<4.75$
$y=6x-18,\ 3.65<x<3.872$
$y-3.5=(x-5.5)^2/20+0.2,\ y<3.87$
$y=-3x+26,\ 3.75<y<4$
$(x-5.8)^2+(y-3.75)^2=0.3,\ x<5.75,\ y<3.7$
$x=5.75,\ 2<y<3.2$
$y=-20x+174,\ 2<y<4$
$y-1.9=(x-7.16)^2/20,\ y<2$
$(x-8.5)^2+(y-4.5)^2=0.1,\ y>4.65$

There is a way to combine all these equations and inequalities into a single equation that produces the same solution set. Can you figure how to do that?

Can you realize one of your favorite drawings as the solutions to a set of equations and inequalities?

Romping Through the Rationals

by Ken Fan | edited by Jennifer Sidney

Jasmine: Emily, you're just the person I want to see!

Emily: What's up?

Jasmine: I learned this curious fact from social media, and I think you're going to love it!

Emily: Something about piano?

Jasmine: Math!

Emily: What is it?

Jasmine: Define $f(x)$ to be the reciprocal of the quantity 1 plus the greatest integer less than or equal to x, less the fractional part of x. That is, let $f(x) = \dfrac{1}{1 + \lfloor x \rfloor - \{x\}}$.

Here, the symbol $\lfloor x \rfloor$ is the "floor" of x, which is the greatest integer less than or equal to x; the symbol $\{x\}$ is the "fractional part" of x, which is equal to $x - \lfloor x \rfloor$.

Emily: That's a peculiar function. What about it?

Jasmine: If you start with 0 then repeatedly apply this function, you will get every single nonnegative rational number exactly once!

Emily: Seriously?

Jasmine: Yes, that's the claim. I haven't figured out how it works yet. Who knows, maybe it's just a joke meme and the claim isn't true. But I want to know.

Emily: So do I![1]

Jasmine: I know that the rational numbers can be put in one-to-one correspondence with the whole numbers, but I've never seen it done so crisply.

Emily: So the claim is that 0, $f(0)$, $f(f(0))$, $f(f(f(0)))$, etc., is a list of all the nonnegative rational numbers?

Jasmine: Yes.

Emily: Let's compute the first few terms. I want to get a feel for this function.

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[1] If you don't get Emily and Jasmine's excitement about this, see page 24.

Jasmine: Starting with $x = 0$, we have $\lfloor x \rfloor = 0$ and $\{x\} = 0$, so $f(x) = 1/(1 + 0 - 0) = 1$.

Emily: For $f(1)$, we have $\lfloor 1 \rfloor = 1$ and $\{1\} = 0$, so $f(1) = 1/(1 + 1 - 0) = 1/2$.

Jasmine: Since $\lfloor 1/2 \rfloor = 0$ and $\{1/2\} = 1/2$, we get $f(1/2) = 1/(1 + 0 - 1/2) = 2$.

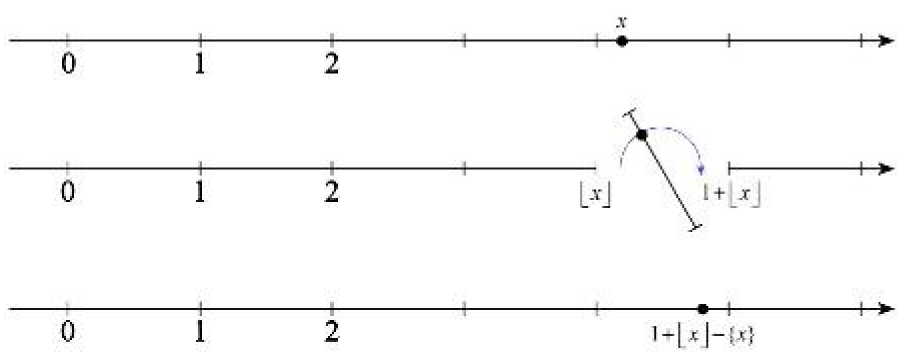
Emily and Jasmine compute the first 19 terms and find

$$0, 1, 1/2, 2, 1/3, 3/2, 2/3, 3, 1/4, 4/3, 3/5, 5/2, 2/5, 5/3, 3/4, 4, 1/5, 5/4, 4/7, \ldots$$

Emily: Amazing!

Jasmine: Maybe it really works!

Emily: While doing the computation, I noticed that the denominator, $1+\lfloor x \rfloor - \{x\}$, has a nice interpretation on the number line. If we plot x on the number line, then $1+\lfloor x \rfloor - \{x\}$ can be found by flipping the interval $[\lfloor x \rfloor, \lfloor x \rfloor + 1]$ around. That's because $1+\lfloor x \rfloor$ is the first integer you encounter (other than x itself) when you move up the number line from x; we then subtract $\{x\}$, which is the amount added to $\lfloor x \rfloor$ to get x.



Emily's geometric interpretation of the denominator of $f(x)$.

Jasmine: That explains why after the 1, the entries of the sequence alternate between being greater than 1 and less than 1; when x isn't 0 or 1, flipping the unit interval that contains x as you described will not change the property of being greater than 1 or less than 1. Reciprocation, however, swaps the numbers greater than 1 with the positive numbers less than 1. So if $x > 1$, then $f(x) < 1$, and if $0 < f(x) < 1$, then $f(x) > 1$.

Emily: Unfortunately, this picture doesn't help me to see that the sequence will contain *all* of the nonnegative rational numbers.

Jasmine: But I think it does imply that no number will appear twice. For if not, let r be the first rational number that appears twice in the sequence. Since the reciprocal of a number is never equal to 0, we know that 0 cannot appear twice, so r is not 0. That means that the first time r appears, it must be preceded in the sequence by some number p, and the second time it appears, it must be preceded in the sequence by some number q. Also, it must be that $p \neq q$, otherwise r wouldn't be the first number that repeats. And that would mean that $f(p) = f(q) = r$, which is the same as saying that f is not one-to-one.

But I think your number line interpretation shows that f *is* one-to-one, because that geometric procedure is reversible, and so is reciprocation.

Emily: Yes, all the steps to get $1+\lfloor x \rfloor - \{x\}$ from x are indeed reversible. So, given $r > 0$, we could reverse the process to find the unique p such that $f(p) = r$. Specifically, we reciprocate r to get $1/r$, and then we can find p by flipping around the interval containing $1/r$ (being sure to take the unit interval to the *left* of $1/r$ in case $1/r$ is an integer).

Jasmine: So at least we know that the sequence can't be periodic, like the decimal digits of a rational number. Now we can focus on showing that *every* nonnegative rational number will appear somewhere in the sequence. Maybe we can try to show that if we reverse the process starting with any positive rational number, we will eventually get to 0?

Emily: That would prove it.

Jasmine: Hmm, I don't see how to show that. It's not like the numbers get closer and closer to 0 as we go backward through the sequence, since the distances that the numbers are from 0 alternate between being greater than 1 and less than 1.

Emily: Actually, I don't think it's possible to list all the rational numbers so that the numbers always increase or always decrease, because there are infinitely many rational numbers greater than any rational number and infinitely many rational numbers between 0 and any positive rational number.

Jasmine: You're right. Any list of the rational numbers is going to have to jump around.

Emily: I'm stuck. I don't know what to do.

Jasmine: Since we didn't come up with the idea, we don't know what motivated it. Well, we can always compute more terms; and since we have no better ideas, why not?

Emily and Jasmine compute more terms:

$$0, 1, 1/2, 2, 1/3, 3/2, 2/3, 3, 1/4, 4/3, 3/5, 5/2, 2/5, 5/3, 3/4, 4, 1/5, 5/4, 4/7,$$
$$7/3, 3/8, 8/5, 5/7, 7/2, 2/7, 7/5, 5/8, 8/3, 3/7, 7/4, 4/5, 5, 1/6, 6/5, 5/9, 9/4, \ldots$$

Emily: Even though the sequence jumps around, it does seem like the nonnegative integers appear in increasing order.

Jasmine: That's curious! I wonder, in what positions do the nonnegative integers appear? Let's see, 0 is the first number, then immediately after that comes 1, then the 2 appears in position 4, and the 3 appears in position … 8. The 4 appears in position … 16. Hey!

Emily: The powers of 2!

Jasmine: It sure looks like the integer n will appear in position 2^n. Let's rewrite the sequence, but this time, after every integer, I'll go to a new line:

> 0,
> 1,
> 1/2, 2,
> 1/3, 3/2, 2/3, 3,
> 1/4, 4/3, 3/5, 5/2, 2/5, 5/3, 3/4, 4,
> 1/5, 5/4, 4/7, 7/3, 3/8, 8/5, 5/7, 7/2, 2/7, 7/5, 5/8, 8/3, 3/7, 7/4, 4/5, 5,
> 1/6, 6/5, 5/9, 9/4, …

Jasmine: It seems convenient to call the first row "row 0"; that way, it appears that row n ends with n.

Emily: Okay. Funny that other than row 0, row n appears to begin with $1/n$.

Jasmine: From the formula, if n is an integer, it's true that $f(n) = 1/(n + 1)$. So when we get to an integer n, we'll start the next line with $1/(n + 1)$. And then … wait a sec!

Emily: Yeah?

Jasmine: It looks like the rows flip if you reciprocate the numbers!

Emily: How do you mean?

Jasmine: For example, look at the row that begins with 1/4 and ends with 4. If we reciprocate the numbers in that row, we get them exactly in reverse order: 4, 3/4, 5/3, 2/5, 5/2, 3/5, 4/3, 1/4!

Emily: I see! That's a cool observation. So if r is a nonnegative rational number, then since r, $f(r)$ are consecutive terms in the sequence, you're saying that $1/f(r)$, $1/r$ would also be consecutive terms in the sequence. That is, you're proposing that $f(1/f(r)) = 1/r$.

Jasmine: Yes … although r cannot be an integer because the flipping occurs only among the numbers in a row, not across rows.

Emily: Let's try to prove this identity.

Jasmine: Okay! Let's write r in lowest terms as P/Q, where P and Q are relatively prime positive integers and Q is not 1. First of all, what is $f(P/Q)$? Let's say that if we divide P by Q we get the quotient K with remainder R, so that $P = QK + R$ and $R < Q$. Then $\lfloor P/Q \rfloor = K$ and $\{P/Q\} = R/Q$. Therefore, $f(P/Q) = 1/(K + 1 - R/Q) = Q/(QK + Q - R)$. Is $Q/(QK + Q - R)$ in lowest terms?

Emily: Let's see. Suppose Q and $QK + Q - R$ have a common divisor D. Then D would have to divide $Q(K + 1) - (QK + Q - R) = R$. But if D divides both Q and R, then D also divides P, since $P = QK + R$. We're assuming that P and Q are relatively prime, so it must be that $D = 1$. Neat! So, yes, $Q/(QK + Q - R)$ is in lowest terms.

Jasmine: Great! Assuming that Q is not 1 (so that r is not an integer), we want to show that $f(1/f(P/Q)) = Q/P$. We just computed $f(P/Q)$, so we know that $1/f(P/Q) = (QK + Q - R)/Q$. To compute $f((QK + Q - R)/Q)$, we need to know what $\lfloor (QK+Q-R)/Q \rfloor$ and $\{(QK + Q - R)/Q\}$ are.

Emily: Well, $(QK + Q - R)/Q = K + 1 - R/Q$. Since R is the remainder we got when we divided P by Q, we know that $0 \leq R < Q$.

Jasmine: In fact, R cannot be 0 and must be positive; if $R = 0$, it would signify that Q divides evenly into P, but that would mean that P/Q is not in lowest terms.

Emily: Oh, right. Then $0 < R/Q < 1$, so $\lfloor (QK+Q-R)/Q \rfloor$ must be K.

Jasmine: And $\{(QK + Q - R)/Q\} = \{K + 1 - R/Q\}$ must be $(Q - R)/Q$.

Emily: I agree. So $f(1/f(P/Q)) = 1/(K + 1 - (Q - R)/Q) = Q/(QK + Q - (Q - R)) = Q/(QK + R)$, but $QK + R$ is just P! So it's true, $f(1/f(P/Q)) = Q/P$!

Jasmine: Cool, we were able to prove something about this function!

Emily: Yes, although I'm not sure if this function identity implies your row-flipping observation, since it's not clear to me that the row that starts with $1/n$ ends with n. Couldn't it end with some other integer?

Jasmine: I'm not sure. But this question of whether the row that begins $1/n$ ends with n, for positive integers n, is equivalent to showing that the integers appear in ascending order. After all, if the row that begins $1/n$ ends with n, the integer $n + 1$ would appear at the end of the next row since $f(n) = 1/(n + 1)$; thus, $n + 1$ would be the next integer that follows n. Conversely, if the integers appear in ascending order, since $1/n$ appears right after the integer $n - 1$, the next integer we encounter after $1/n$ would have to be n. But how are we going to show either fact?

Emily: I think I may see something …

To be continued …

Examples of Posets

by Robert Donley[1]
edited by Amanda Galtman

In previous installments of this series, path counting methods led to important classes of integers, such as binomials coefficients, Catalan numbers, and Fibonacci numbers. Convolution also appeared as a theme in various forms: Chu-Vandermonde convolution, Segner's recurrence, and the Cauchy product for sequences. Now we generalize our path model in a way that extends our approach to one of the most important techniques in combinatorics, partial orderings.

Some years ago, we hosted a chocolate tasting at Girls' Angle.[2] Sometimes, a girl had a definite preference for one chocolate over another. But sometimes, two chocolates were incomparable. Which one tasted better? Neither; they were equally delicious, but definitely distinct in their flavor. Chocolate preferences can provide examples of what's known as a **partially ordered set**. We're only partially able to order the chocolates. If we use the symbol $\leq$ to mean "is not as yummy as or is the same as," for any two chocolates x and y, it may be that $x \leq y$ or $y \leq x$, but it could also be that x and y are incomparable. However, it is still true that every chocolate "is not as yummy as or is the same as" itself, and if you have three chocolates x, y, and z such that $x \leq y$ and $y \leq z$, then it is surely the case that $x \leq z$. What happens if $x \leq y$ and $y \leq x$? It can't be that x is not as yummy as y while at the same time y is not as yummy as x, so it must be that x is the same as y.

These considerations lead to this formal definition of a partially ordered set.

Let P be a set. First, we define a **relation** on P to be a set of ordered pairs (x, y) with x and y in P. If (x, y) is in the relation, we say that "x is related to y." For brevity, we sometimes give the relation a name, such as R, and then write "xRy" when x is related to y.

Definition. A set P with relation R is called a **partially ordered set** (or **poset** for short) if the relation satisfies the following three properties:

- Reflexivity: for all x in P, we have xRx.
- Anti-symmetry: for all x and y in P, if xRy and yRx, then $y = x$.
- Transitivity: for all x, y, and z in P, if xRy and yRz, then xRz.

Exercise: The P be the set of integers and let R be the relation corresponding to the usual $\leq$. (That is, the relation is the subset of ordered pairs (x, y) where x and y are integers and $x \leq y$.) Verify the three properties of reflexivity, anti-symmetry, and transitivity.

With the less-than-or-equal-to relation, the integers are not just partially ordered. They are **totally ordered**, because in addition to the three properties listed above, for any two integers, one must be less than or equal to the other. There are no incomparable pairs of integers. However, in a partially ordered set, we can have elements in the set that are *not* comparable under the relation. For example, consider the set of subsets of the set $\{1, 2\}$, and introduce a partial order by saying that for subsets S and T of $\{1, 2\}$, we have $S \leq T$ if and only if S is a

[1] This content is supported in part by a grant from MathWorks.
[2] See page 9 of Volume 2, Number 2 of the Girls' Angle Bulletin.

subset of T. In this example, notice that the subsets $\{1\}$ and $\{2\}$ are incomparable because neither is a subset of the other.

When no confusion arises, we denote a relation that corresponds to a partial order by $\leq$. We borrow the symbol for "less than or equal to" even though the situation may have nothing to do with comparing numbers.

Definition. We say x is **covered** by y if x and y are distinct elements such that $x \leq y$ and there are no elements between x and y. (That is, there is no z distinct from x and y such that $x \leq z \leq y$.)

Definition. The **Hasse diagram** of P (or simply **diagram** of P) with respect to $\leq$ is the diagram such that

- Each element of P corresponds to a node in the diagram,
- a link is drawn if x is covered by y, and
- the diagram is oriented such that smaller elements appear below larger elements.

Example: Let $P = \{a, b, c, d\}$. We can turn P into a partially ordered set by defining the relation $\leq$ on P. We declare that $a \leq b$, $a \leq c$, $b \leq d$, $c \leq d$, and $a \leq d$. We further declare that $a \leq a$, $b \leq b$, $c \leq c$, and $d \leq d$. (Please verify that this relation satisfies the three defining properties needed to be a partial order.) The Hasse diagram of P is shown in Figure 1.

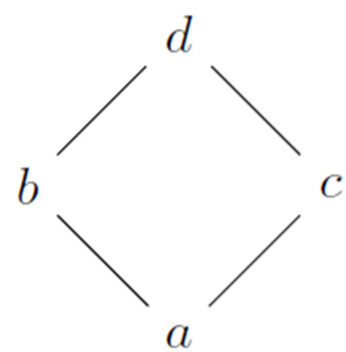


Figure 1. A Hasse diagram.

If we know all covering pairs $x \leq y$, we can reconstruct the partial order. The transitivity property lets us link consecutive covering relations; for instance, by transitivity, $a \leq b$ and $b \leq d$ imply $a \leq d$. In the diagram, we see this implication as a sequence of connected links between the smaller element to the larger one.

Definition. When $x \leq y$, a **saturated chain** from x to y in the diagram of P is a sequence of links of consecutive covering relations. The **length** of a saturated chain is the number of links. Since all chains considered here are saturated, we shall use the word "chain" to mean a saturated chain.

For instance, in the above example, $\{a \leq b, b \leq d\}$ is a chain from a to d of length 2.

In P, note that b and c are not related. Neither $b \leq c$ nor $c \leq b$. In this case, we say that b and c are **incomparable**.

Example: Suppose $P = \{0, 1, 2, 3, ..., n\}$ for some positive integer n, where $\leq$ corresponds to the usual less-than-or-equal-to relation. Then P is a chain of length n.

Example: For nonnegative integers m and n, consider the rectangle in the xy-plane with corners at $(0, 0)$, $(m, 0)$, $(0, n)$ and (m, n). Let $P_{m,n}$ be the set of lattice points contained on or inside this rectangle. That is, let $P_{m,n}$ be the set of ordered pairs (r, s) with $0 \leq r \leq m$ and $0 \leq s \leq n$. We now introduce a partial order on $P_{m,n}$ by describing its Hasse diagram. Draw all the horizontal and vertical unit length line segments between adjacent lattice points in $P_{m,n}$. Obtain the Hasse diagram of $P_{m,n}$ by rotating this drawing by 45 degrees counterclockwise. (Figure 1 illustrates the specific case $m = n = 1$ if we identify a with $(0, 0)$, b with $(0, 1)$, c with $(1, 0)$, and d with $(1, 1)$.)

Exercise: Let (r, s) and (x, y) be in $P_{m,n}$. In terms of r, s, x, and y, when does (x, y) cover (r, s)?

Exercise: How many covering relations are there in this partial order for $P_{m,n}$? How many chains are there from $(0, 0)$ to (m, n)? If, at each element of $P_{m,n}$, you write down the number of chains from $(0, 0)$ to (m, n), do you recognize the resulting arrangement of numbers?

Example: Fix a positive integer n. Let's modify the previous example by letting C_n consist of the points (r, s) where $0 \leq s \leq r \leq n$. Form the Hasse diagram in the same way we did for $P_{m,n}$, but restrict to this isosceles right triangle of points. See Figure 2 for an illustration of the Hasse diagram of C_3.

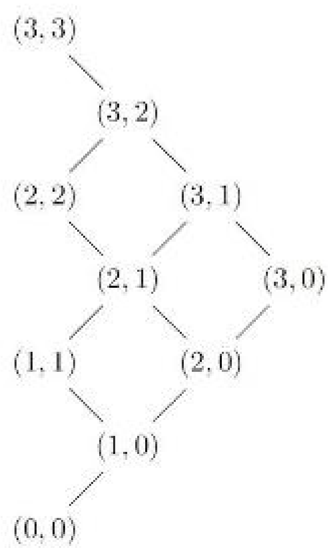


Figure 2. The Hasse diagram of C_n when $n = 3$.

Exercise: How many nodes are in the Hasse diagram of C_n? How many covering relations are there? How many chains are there from $(0, 0)$ to (n, n)? Do you recognize the sequence of numbers whose kth number is the number of chains from $(0, 0)$ to (k, k) in C_n?

Example: Let H be the set of points in 3D space whose coordinates (x, y, z) are nonnegative integers. Analogously to the construction of the Hasse diagram of $P_{m,n}$, we can describe all the covering relations in H by saying that for any nonnegative integers x, y, and z, each of the points $(x + 1, y, z)$, $(x, y + 1, z)$, and $(x, y, z + 1)$ covers (x, y, z). We define the **height** of the point (x, y, z) to be $x + y + z$.

Exercise: How many points are there in H with a given height h?

To count the number of chains from (0, 0, 0) to (m, n, p), where m, n, and p are nonnegative integers, we can count words made up of three letters, such as F (forward), R (rightwards), and U (up). A word corresponding to a chain from (0, 0, 0) to (m, n, p) has m F's, n R's, and p U's. Let $h = m + n + p$. Then there are $C(h, m)$ choices for the positions of F in the word. After those are chosen, there are $C(h - m, n)$ choices for the positions of R in the word. And once the positions of all the F's and R's are chosen, remaining positions must be the letter U. By the Matching Rule, the number of words is

$$C(h, m)C(h-m, n) = \frac{h!}{m!(h-m)!} \cdot \frac{(h-m!)}{n!p!} = \frac{(m+n+p)!}{m!n!p!}.$$

Exercise: Reformulate the above example and calculations for 4-tuples of nonnegative integers instead of 3-tuples.

If we restrict H to the set of points whose coordinates are 0 or 1, we obtain an example of a **Boolean poset**. In general, let n be a positive integer and let B_n be the set of n-tuples with entries 0 or 1. We turn B_n into a partially ordered set by declaring that the n-tuple x is less than or equal to the n-tuple y if and only if each coordinate of x is less than or equal to the corresponding coordinate of y. See Figure 3 for the Hasse diagram of B_3. Notice that points are at the same height in a Boolean poset if and only if they have the same number of coordinates that are 1.

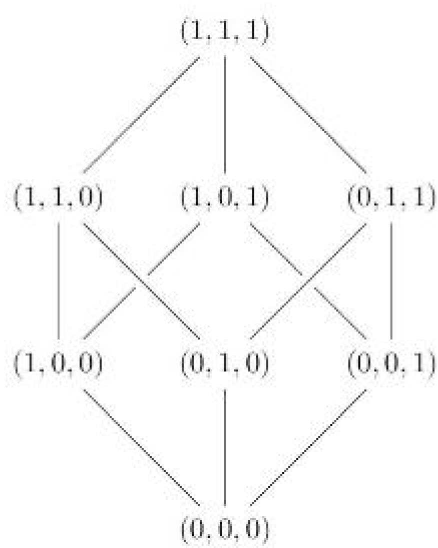


Figure 3. The Hasse diagram of the Boolean poset B_3.

Exercise: Draw the Hasse diagram of B_4. How many elements are in B_n? How many elements are at height h in B_n? Describe how to form a chain from (0, …, 0) to (1, …, 1) in B_n.

Suppose X is a set with n elements labeled $x_1, x_2, x_3, \ldots, x_n$. The power set of X, denoted $P(X)$, is the set of all subsets of X. We can make $P(X)$ a partially ordered set if, for any subsets U and V of X, we declare that $U \leq V$ if and only if U is a subset of V.

In fact, we've already seen the poset $P(X)$, because it's equivalent to B_n!

Exercise: Show how the elements of B_n can be matched with elements of $P(X)$ in such a way that whenever x is less than or equal to y in B_n, the element corresponding to x in $P(X)$ is a subset of the element that corresponds to y, and vice versa.

(Spoiler Alert!) There's more than one way to do the previous exercise. We'll match elements of $P(X)$ with elements of B_n by representing the subset U of X by the n-tuple whose ith coordinate is 0 if x_i is not in U and 1 if x_i is in U. Thus, the element (0, 0, 0, …, 0) corresponds to the empty set.

Note that the subset U is covered by the subset V if V is obtained by adding one new element to U.

Exercise: Let U be a subset of X. How many chains are there from the empty set to U?

Exercise: List all six chains from the empty set to $X = \{1, 2, 3\}$ in $P(X)$. Also, list the corresponding chains in B_3.

To obtain a general version of Chu-Vandermonde convolution for posets, we first note some common features in our examples of posets with finitely many elements (which do not necessarily occur in other finite posets):

- There exist both minimum and maximum elements. Call them m and M. Then every x in P satisfies $m \le x \le M$. In other words, every element x occurs in some chain from m to M.
- The poset P is **graded**, that is, the length of any chain from m to M is the same.
- The graded property allows us to define the **rank** of any element x in P, which is the length of any chain from m to x. (The rank is what we called the height in the posets H and B_n.)
- The collection of all elements with rank h is called the hth **level** of P and is denoted by P_h. The graded property of our poset ensures that links cannot skip levels.

We can now repeat the argument for the Chu-Vandermonde convolution formula in the poset context. For a fixed height h, every chain from m to M passes through a unique element of P_h, and all such chains pass through some element of P_h. Therefore, to count all chains from m to M, we can count the number of chains through each element of P_h and sum these counts. For a fixed x in P_h, the number of such chains is equal to

$$(\text{number of chains from } m \text{ to } x) \times (\text{number of chains from } x \text{ to } M).$$

Example: For the Boolean poset B_n with $0 \le h \le n$, an element x of rank h has exactly h ones, and there are $h!$ chains from $m = (0, ..., 0)$ to x. Likewise, the number of chains from x to the maximal element $M = (1, ..., 1)$ is $(n - h)!$, so there are $h!(n - h)!$ chains through x. This count is the same for all elements of P_h. Since the number of elements of rank h is $C(n, h)$, we verify that the total number of chains from m to M in B_n is given by

$$C(n, h)h!(n - h)! = n!.$$

Example: We use the chain counting formula to obtain the family of identities

$$F_mF_n + F_{m+1}F_{n+1} = F_{m+n+1},$$

where F_k is the sequence of Fibonacci numbers (with $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for all $n > 1$). The special case when $m = n$ was given in the previous installment in this series. Consider the poset of nodes (x, y) in $P_{n,n}$ such that $x - 2 \le y \le x + 1$. We call this the "Fibonacci poset." Figure 4 shows the Fibonacci poset in the case where $m = n = 3$, alongside the various chain counts from the minimal element (0, 0) to the various elements.

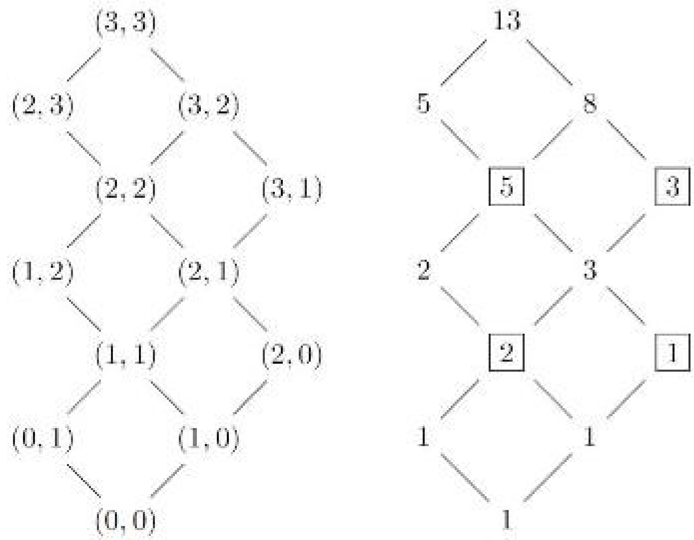


Figure 4.

Notice that the Fibonacci poset has vertical mirror symmetry. By applying the Chu-Vandermonde convolution formula for counting chains from (0, 0) to (3, 3), we get a number of identities, depending on which level we apply the formula to. From Figure 4, we get the following identities:

$$13 = 1 \cdot 5 + 1 \cdot 8 = 1 \cdot 3 + 2 \cdot 5 = 2 \cdot 2 + 3 \cdot 3,$$

i.e.,

$$F_7 = F_1F_5 + F_2F_6 = F_2F_4 + F_3F_5 = F_3F_3 + F_4F_4.$$

Exercise: Explain how the Fibonacci numbers give the number of chains from the minimal element to the various elements in these Fibonacci posets.

Exercise: Explain the identities $F_mF_n + F_{m+1}F_{n+1} = F_{m+n+1}$ in terms of Chu-Vandermonde convolution for counting chains in these Fibonacci posets.

Exercise: Repeat the above analysis for the Catalan numbers using posets.

The Needell in the Haystack[1]

Classification and the Brain: The Perceptron Algorithm
by Anna Ma | edited by Jennifer Sidney

What do you think of when you hear the words "artificial intelligence"? Perhaps you imagine a realistic robot that can walk and talk like a human. Or maybe you picture a house that cleans your room for you, knows when you're hungry and makes you a snack, and finds your phone when you've misplaced it. In each of these examples, a machine or computer is trained to perform tasks that a human would/can do: walking, talking, preparing food, and so forth. These applications sound nice, but how in the world can we expect anything to be able to learn all these tasks? Is this even feasible? Actually, it already happens: our *brains* can do all of these tasks and much, much more. If we want to create artificial intelligence, why not start by trying to model the brain?

Your brain is composed of billions of nerve cells called **neurons**. Each of these neurons is connected to thousands of other neurons creating a network of neurons, or a **neural network**. Very broadly, your neural network works in the following way: you take environmental inputs, process those inputs through your neural network (i.e., brain), and output a response. For example, when you see and smell your favorite pizza, these visual and olfactory signals are sent to your brain, which then triggers an output response that tells your mouth to salivate to prepare for digestion. Your brain does this so quickly that you probably don't even think about it!

It's natural to ask whether we can computationally model our brain's neural network, essentially giving a computer a human-modeled brain. The simple answer is this: not yet. The brain is extremely complex. Computer scientists, mathematicians, statisticians, cognitive scientists, and so forth still do not have a full understanding of how to accomplish such a feat. To build up to this goal, we might start by trying to teach a computer to perform a simple task known as **binary classification**, which uses a single computational neuron.

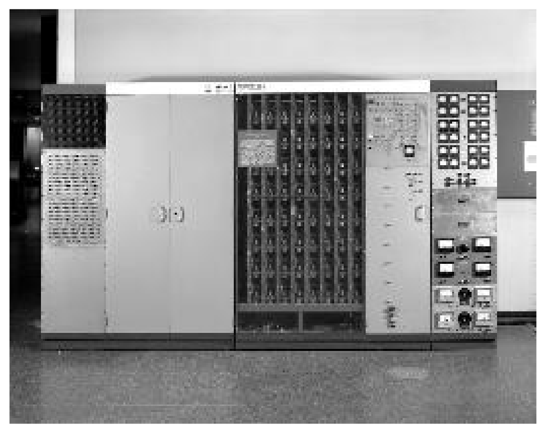


Figure 1. This machine, called the Mark 1 Perceptron, was used to model a single neuron in the 1950s. It had a camera attached to it and was trained to classify pictures of men and women.

In machine learning, binary classification – the task of classifying data points into two sets – is heavily relied on for a multitude of tasks. Spam filters try to classify whether an email is spam or not. Facial recognition algorithms on phones try to determine whether a person is the owner of the phone or not. For these tasks, data is used to train an algorithm to perform the task. In this installment of "The Needell in the Haystack," we will present an algorithm that can be used for binary classification: the **perceptron algorithm**. It is a supervised machine learning algorithm that can be used to find a linear separator for labeled data.

[1] This content is supported in part by a grant from MathWorks. Anna Ma is a Visiting Assistant Professor at the University of California Irvine.

Figure 1 shows a picture of one of the first implementations of a perceptron,[2] which was designed to emulate a single neuron in our brain. This machine had a camera attached to it. Scientists would show pictures of people through the camera, then indicate to the machine whether the picture was of a man or a woman. Similar to a simplified model of a neuron, the perceptron takes an input, processes that input, and then provides an output. You might notice that this machine is huge! Nowadays, we don't need such a huge machine to run a perceptron algorithm. In fact, it can be implemented with just a few lines of code on a smartphone. Before we can discuss the algorithm used to train the perceptron, let's revisit linear classifiers.

Recap: Linear Separators and Classifiers

Let $a = (a_1, a_2, \ldots, a_d) \in \mathbb{R}^d$ denote a data point and $y \in \{-1, +1\}$ be its label such that if $y = +1$, then we will say that the data point a belongs to class +1 and if $y = -1$, we will say that the data point a belongs to class -1.

Given data points and their labels, our goal in binary classification is to **separate** $\mathbb{R}^d$, the space in which our data points exist, into two regions: one region containing all the data points labeled +1, and one region containing all the data points labeled -1. The object that separates these points in space is called the **separator**, and we call a separator that is linear a **linear separator**. In $\mathbb{R}^2$, a linear separator is a line and in $\mathbb{R}^3$, it's a plane. Figure 2 shows examples of linear separators in 2 and 3 dimensions, referred to in these images as **hyperplanes**. In higher dimensions, hyperplanes become more difficult to visualize; mathematically, though, they can all be described as the points $(x_1, x_2, \ldots, x_d)$ which satisfy a linear equation of the form $w_1x_1 + w_2x_2 + \ldots + w_dx_d = v$, where the w_i and v are constants and not all the w_i are equal to 0.

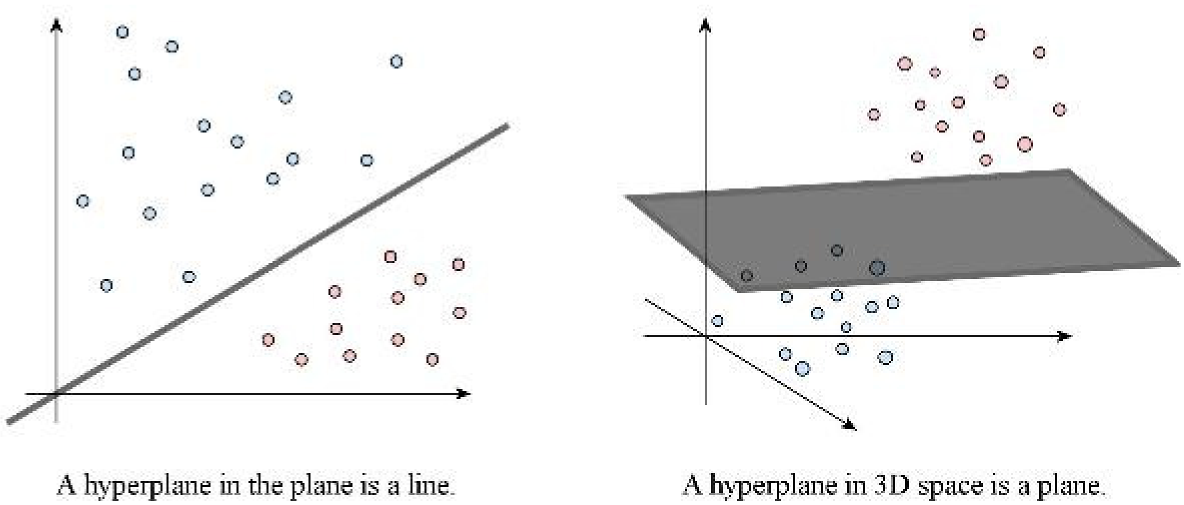


Figure 2. Examples of hyperplanes in 2 and 3 dimensions.

To clarify the key features of the algorithm we are about to describe, we shall assume that our linear separator contains the origin. (This restriction can be overcome using the same algorithm, but adding a dimension and putting a 1 in the coordinate for that dimension in all the data points.) In other words, we will consider hyperplanes of the form

$$w_1x_1 + w_2x_2 + \ldots + w_dx_d = 0. \tag{1}$$

[2] There's also a neat video of it available here: www.youtube.com/watch?v=cNxadbrN_aI.

Using (1) to determine our linear separator, we can define a **linear classifier**. For 2D data, a linear classifier is a function $f(x_1, x_2) : \mathbb{R}^2 \to \{-1, +1\}$ that outputs a label based on a linear separator. The function

$$f(x_1, x_2) = \begin{cases} +1 & \text{if } w_1x_1 + w_2x_2 \geq 0 \\ -1 & \text{if } w_1x_1 + w_2x_2 < 0 \end{cases} \tag{2}$$

is a linear classifier, as it assigns an input data point $a = (a_1, a_2)$ to a class label of -1 or +1.

For d-dimensional data, this classifier can be equivalently written as

$$f(x) = \begin{cases} +1 & \text{if } \langle w, x \rangle \geq 0 \\ -1 & \text{if } \langle w, x \rangle < 0, \end{cases} \tag{3}$$

where $x, w \in \mathbb{R}^d$ and $\langle w, x \rangle = \sum_{i=1}^{d} w_i x_i$ denotes the inner product.

The Perceptron Algorithm

The perceptron algorithm is an iterative method that finds the linear separator defined by the parameters w_1 and w_2 in (2). It does so by checking and correcting: the algorithm makes a guess of what it thinks the linear separator should be, then uses one of the points in the data set to check whether it's correct. If it's correct, it then moves on to another point. If it's not, then it updates the separator so that it's correct (for that and previous data points). Let's watch this algorithm at work!

Consider the following data points, each with its associated label placed underneath:

(-2, -2)	(-3, 0)	(-3, 3)	(-2, -1)	(-2, 0)	(-3, -3)	(1, -4)	(1, -1)	(1, 3)	(2, 1)	(3, -1)	(3, -2)
-1	-1	-1	-1	-1	-1	-1	+1	+1	+1	+1	+1

For example, data point (-3, 3) has label -1 and data point (3, -1) has label +1.

When we start the algorithm, we assume no knowledge about the data or labels. Since we have no information, we can randomly choose a line to start with. Choosing $w_1 = -1$ and $w_2 = 0$, our initial linear separator is $x_1 = 0$, as shown in the left subfigure of Figure 3. Since this is our initial guess for the vector $w = (w_1, w_2)$, we denote it $w^0 = (-1, 0)$, and we'll continue to use the notation w^t to denote the t-th approximation of w.

Next, we look at the label and coordinates of one of our data points. As shown in the middle subplot of Figure 3, we have selected the point $a = (-2, 2)$ with associated label $y = -1$. This data point is *incorrectly labeled* by our current classifier Eq. (2) with $w = w^0 = (-1, 0)$. Since

$$\langle w^0, a \rangle = \langle (-1, 0), (-2, 2) \rangle = 2 + 0 = 2,$$

the linear classifier assigns data point (-2, 2) a label of +1, whereas its correct label is $y = -1$. Thus, we want to update the linear separator based on this new information so that $f(a) = -1$. We

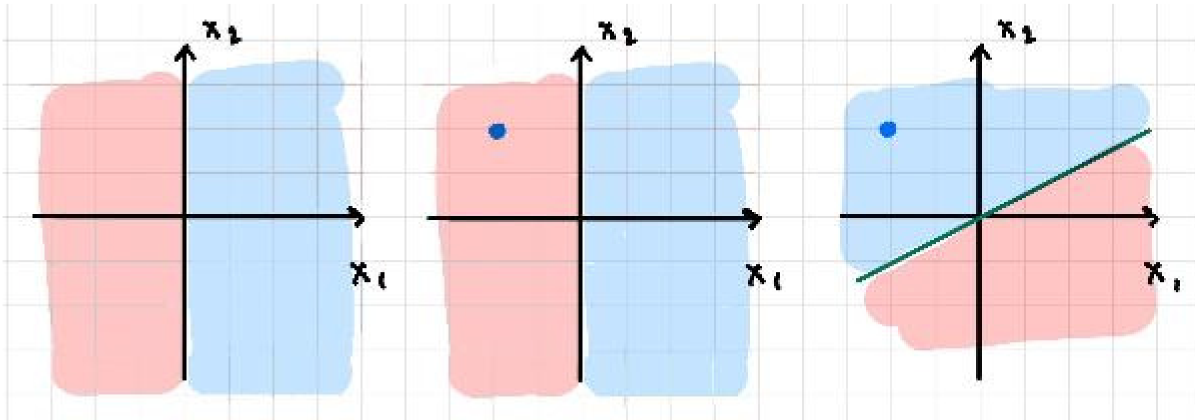


Figure 3. One iteration of the perceptron algorithm starting with initial space (left), checking a point (middle), and updating the linear separator (right).

started with the vector $w^0 = (-1, 0)$. To update w, the perceptron algorithm uses a and its label y to create the next approximation to w: $w^1 = w^0 + ya = (-1, 0) + (-1)(-2, 2) = (1, -2)$. This ensures that the point that was previously incorrectly labeled is now correctly labeled, since

$$\langle w^1, a \rangle = \langle (1,-2), (-2,2) \rangle = -2 - 4 = -6 < 0\,.$$

With $w^1 = (1, -2)$, Eq. (2) now assigns the point (-2, 2) a label of -1, as desired. To plot the new linear classifier, we plot the line corresponding to $x_1 - 2x_2 = 0$ as shown in the right subplot of Figure 3.

Now consider a data point we haven't used: $a = (1, 3)$. This point is associated with the label $y = +1$, but our current classifier misclassifies this point. We can verify this by checking

$$\langle w^1, a \rangle = \langle (1,-2), (1,3) \rangle = 1 - 6 = -5 < 0\,.$$

Thus, the linear separator with $w^1 = (1, -2)$ incorrectly assigns a label of -1 to our new point. This can also be seen in the left subplot of Figure 3 since the new point x (red star) falls into the blue region instead of the red region. To address this, we perform the same update. Starting with $w^1 = (1, -2)$, the perceptron algorithm produces a new approximation for w:

$$w^2 = w^1 + ya = (1, -2) + (1)(1, 3) = (2, 1).$$

This ensures that the new point will have a label of +1, since

$$\langle w^2, a \rangle = \langle (2,1), (1,3) \rangle = 2 + 3 = 5 \geq 0\,.$$

The new linear separator, the line $2x_1 + x_2 = 0$, is shown in the middle subplot of Figure (3). In the right subplot of Figure (3), we see that all points are classified correctly now.

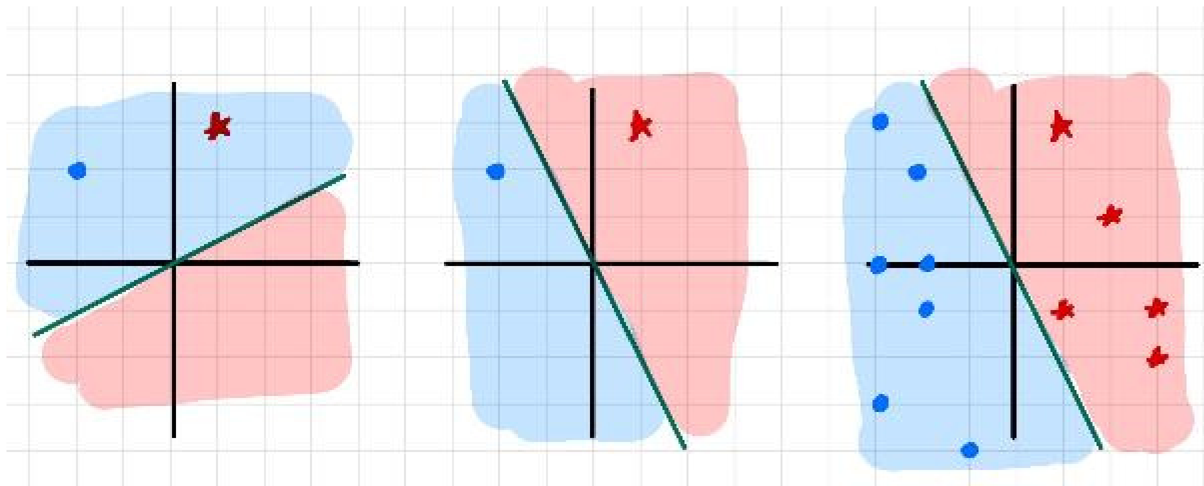


Figure 4. One iteration of the perceptron algorithm checking a new point (left), updating the linear separator (middle), and then checking the remaining points (right).

Notice that with each new misclassified data point, we update our approximation of w. This is what makes the perceptron algorithm an **iterative method**. Iterative methods produce a sequence of steps that aim to improve the approximate solution to a problem. The perceptron algorithm can be written as follows:

Step 1. Initialize the vector $w^0 \in \mathbb{R}^d$ and set $t = 0$.

Step 2. For each data point a and its associated label y in the data set:

(a) Calculate the label $f(a)$ predicted by the current classifier using $w = w^t$ in Eq. (3).
(b) If the classifier is wrong, i.e., if $f(a) \neq y$, then let $w^{t+1} = w^t + ya$. If the classifier is correct, let $w^{t+1} = w^t$. Increment t.

If our data cannot be separated by a linear separator, this perceptron algorithm will not converge. However, if the data can be separated by a linear separator, then the algorithm will find a linear separator!

Now that the algorithm is laid out for you, you can try to implement this algorithm by hand or, if you're familiar with coding, in a coding language of your choice. One fun aspect of numerical algorithms such as the perceptron algorithm is that we can experiment with different variations of it. Try changing the order of the data. Does this significantly impact the end result?

Computational neural networks are still of great interest to scientists and mathematicians, both theoretically and from a practical standpoint. Neural networks are used to train self-driving cars, for image processing and curation, and much more. The linear classification problem is just the beginning; there's still so much more to explore!

Learn by Doing

Listing the Rational Numbers
by Addie Summer

If you're unsure why Emily and Jasmine (see page 8) are so excited about the way to list the nonnegative rational numbers that they learned on social media, this Learn by Doing is for you.

Rational Numbers

A **rational number** is a number that can be expressed as the ratio of two integers.

For example, 1/2, 5/3, and 7 are all rational numbers. (Note that 7 can be expressed as 7/1.)

We shall denote the set of rational numbers by **Q**.

1. Show that a rational number can be expressed as a ratio of integers in more than one way. In fact, every rational number can be expressed as a ratio of integers in infinitely many ways.

There are infinitely many rational numbers. There are many ways to see this fact. For example, one way is to note that the rational numbers contain the integers, and there are infinitely many integers, so there must be infinitely many rational numbers.

2. For another way, show that the arithmetic mean of two rational numbers is, again, a rational number. In particular, if s and t are distinct rational numbers, and $a = (s + t)/2$ is their arithmetic mean, then a is between s and t. That is, between any two distinct rational numbers, there is a rational number. How does this imply that there are infinitely many rational numbers?

3. In fact, show that between any 2 distinct rational numbers, there are *infinitely many* rational numbers.

Because there are infinitely many rational numbers, they enjoy some properties that wouldn't make sense for a finite set of numbers.

4. Let F be a finite set of numbers that contains at least 2 numbers. Let $T = \{2x \mid x \text{ is in } F\}$, that is T is the set of numbers that are twice the numbers in F. For example, if $F = \{-1, 1, 4\}$, then $T = \{-2, 2, 8\}$.

Why Rational Numbers?

You have 3 identical cakes and need to share it among 5 people. How much cake should each person get? The answer is described by a rational number. Conceptually, we know that the answer is an amount of cake that allows each of the 5 people to get the same, maximum possible, amount of cake. If we let A be this amount of cake, then 5 times A is the total amount of cake: $5A = 3$ cakes. To solve for A, we divide both sides of this equation by 5 and get $A = 3/5$ cake, an answer that involves a ratio of integers. It would be convenient if 5 divided evenly into 3, because that would mean each person would get a whole number of cakes and you wouldn't have to cut up the cakes into smaller pieces to share them. But 5 does not divide evenly into 3, and so we have to work with the rational number 3/5.

Numbers are often born out of the desire to be able to provide solutions to some equation. If all you had were the counting numbers (1, 2, 3, 4, ...), then the equation

$$5 + x = 3$$

would not have a solution. To provide solutions to such equations, you can expand the set of numbers to include negative numbers (..., -3, -2, -1, 0, 1, 2, 3, ...). These numbers are the **integers**. But if all you had were the integers, then the equation

$$2x = 5$$

would not have a solution. To provide solutions to such equations, you can expand the set of numbers again to include ratios of integers. That's exactly what the rational numbers are.

If all you had are rational numbers, then the equation $x^2 = 2$ would not have a solution. How would you expand the rational numbers so that this equation has a solution?

Show that F and T are different sets. In fact, show that there is a number in F which is not in T and a number in T which is not in F. (Why did we require that F contain at least 2 numbers?)

5. However, let S be the set of numbers that are twice a rational number. That is, define S to be the set $\{2x \mid x \text{ is in } \mathbf{Q}\}$. Show that $S = \mathbf{Q}$.

6. (This problem is a detour from the main thread of this Learn by Doing problem set.) Instead of defining S as in Problem 5, consider $S = \{x^2 \mid x \text{ is in } \mathbf{Q}\}$. Show that S is a proper subset of $\mathbf{Q}$. (A subset of a set is proper if it does not contain all the elements of the set.)

It is well-known in math circles that one can make a list (albeit, an infinite one) of all the rational numbers. Another way of saying this is that there is a one-to-one correspondence between the set of positive integers and the set of rational numbers. If you know about **cardinality**, this is the same as saying that the rational numbers have the same cardinality as the set of positive integers. (If you have two infinite sets, it is not always possible to establish a one-to-one correspondence between the elements of the two sets.)

A great way to gain appreciation for something is to try to do it yourself. With that in mind,

7. Try to make a list of the positive rational numbers. For this problem, we don't mean for you to explicitly write down a list, because that would be impossible! What we mean is for you to describe what the list is or how it is constructed precisely enough that you can prove that your list contains all the positive rational numbers each exactly once. For example, if we were asked to do this problem for the positive even integers, we might answer as follows: We list the positive even integers as follows: 2, 4, 6, 8, 10, 12, ..., where the kth entry in the list, where k is a positive integer, is the even integer $2k$. To see that every positive even integer is in this list, note that if n is an even positive integer, then n is evenly divisible by 2, that is $n/2$ is an integer. If we let $m = n/2$, then $n = 2m$. By construction, the mth integer in our list is $2m$. Hence, $n = 2m$ is in our list. Also, note that if $2k = 2j$, then $k = j$, hence no even integer appears twice in our list.

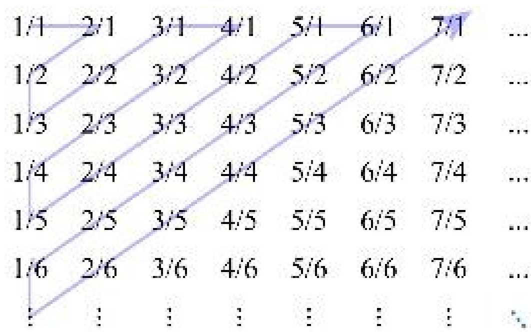
Let's look at a classic solution to Problem 7.

We imagine an infinite array whose first row consists of all ratios of integers, in ascending order, with a denominator of 1, whose second row consists of all ratios of integers, in ascending order, with a denominator of 2, whose third row consists of all ratios of integers, in ascending order, with a denominator of 3, etc.

1/1	2/1	3/1	4/1	5/1	6/1	7/1	...
1/2	2/2	3/2	4/2	5/2	6/2	7/2	...
1/3	2/3	3/3	4/3	5/3	6/3	7/3	...
1/4	2/4	3/4	4/4	5/4	6/4	7/4	...
1/5	2/5	3/5	4/5	5/5	6/5	7/5	...
1/6	2/6	3/6	4/6	5/6	6/6	7/6	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

It won't work to try to make a list of the rational numbers by going through the fractions in this array row by row since each row has infinitely many fractions, so we would never get to the end of the first row.

However, instead of going through the entries row by row, we traverse them diagonally like this:



Note that as we travel through the array, we will encounter each rational number several times. For example, the number 1 is met when we pass through the fractions 1/1, 2/2, 3/3, 4/4, etc.

8. Explain why every positive rational number is in the array.

9. We form a list by writing down the fractions we encounter when we travel through the array diagonally as indicated. If we come upon a number we have already visited before, we simply skip over that number. Explain why every rational number will appear in some finite position in our list.

While this classic solution does show us how to make a list of the positive rational numbers, it is unsatisfying in some ways. For one thing, it's not clear where in the list a specific rational number will occur. For example,

10. In which position does the number 99/100 occur in this list? (We don't actually intend that you figure this out. The intention is only that you make an attempt so that you realize the difficulty of determining the answer to the question.)

(If you really must know, 99/100 is the 12,053rd number in this list!)

In the method of listing the nonnegative rational numbers that Emily and Jasmine are investigating, the two already conjecture that the integer n will be the 2^n-th term in the list. By contrast, using the method described above, here are the positions of the numbers 1 through 20:

1, 2, 5, 6, 11, 12, 21, 22, 31, 32, 45, 46, 63, 64, 79, 80, 101, 102, 127, 128.

It's not at all clear how to determine the next terms of this sequence without going to the trouble of making the list. However, if you're up for a big challenge, there is this:

11. Let $p(n)$ be the position of n in the list of positive rational numbers described above (and whose first 20 terms were just provided). Show that $2p(n)/n^2$ converges to $6/\pi^2$.

12. What's the relationship between this topic and the cover of this magazine?

Notes from the Club

These notes cover some of what happened at Girls' Angle meets. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

Session 31 - Meet 1 September 8, 2022	Mentors:	Elisabeth Bullock, Jade Buckwalter, Cecilia Esterman, Tharini Padmagirisan, AnaMaria Perez, Vievie Romanelli, Emma Wang, Jane Wang, Rebecca Whitman, Muskan Yadav

Excellent communication is crucial to the way in which we handle math education, and much communication is done via subtle body language. For one thing, it's much easier to tell if someone is thinking or disengaged if you can see them, but if their camera feed is turned off and they are muted in a virtual meeting, there's no way to tell the difference without asking. However, asking carries the risk of interrupting the thought process if the person is thinking. For reasons such as this, the virtual years of Girls' Angle were difficult. Thanks to Google Cambridge, we are back meeting in-person! Goodbye mute button, hello free-flowing conversations!

All of us at Girls' Angle thank Google Cambridge for hosting us, and especially to Google employees Cammie Smith Barnes and Andrew Harteveldt, and all those who have been or will be chaperoning our meets there.

Session 31 - Meet 2 September 15, 2022	Mentors:	Jade Buckwalter, Tharini Padmagirisan, Kate Pearce, AnaMaria Perez, Laura Pierson, Vievie Romanelli, Emma Wang, Jane Wang, Rebecca Whitman, Muskan Yadav

Although we strive to put our members into the driver's seat of their own math education, the process of getting there can take some time. In order to encourage members to generate thoughts, we will sometimes engage them with a math game whose rules practically beg for modification. One can learn a lot about where a person is in math by observing the rules that they invent.

Session 31 - Meet 3 September 22, 2022	Mentors:	Cecilia Esterman, Jenny Kaufmann, Kate Pearce, AnaMaria Perez, Laura Pierson, Jing Wang, Rebecca Whitman, Muskan Yadav

The concept of the greatest common divisor leads to a lot of mathematics, and nearly half of our members are working on math where this concept plays a role. At this early stage, these members form a number of groups that are not aware of what the others are working on. But it won't be long before they discover each other. Who knows what synergies might result?

On this issue's cover, the greatest common divisors of pairs of positive integers for several pairs is depicted. What patterns do you see?

Let $Q = \{(x, y) \mid x \text{ and } y \text{ are positive integer}\}$. For any subset S of Q, define nS to be the set of ordered pairs (nx, ny), where (x, y) is in S. Let G be a subset of Q with the following property: Q is the disjoint union of nG, over all positive integers n. What can you say about G?

Session 31 - Meet 4 September 29, 2022 | Mentors: Elisabeth Bullock, Jade Buckwalter, Cecilia Esterman, Tharini Padmagirisan, Laura Pierson, Jane Wang, Jing Wang, Rebecca Whitman

Some members learned to fold Fujimoto's hydrangea, an origami model that involves only horizontal, vertical, and 45°-diagonal folds, producing a model that could, in theory, be folded ad infinitum. The model illustrates similarity and geometric series nicely.

Session 31 - Meet 5 October 6, 2022 | Mentors: Jade Buckwalter, Cecilia Esterman, Tharini Padmagirisan, Kate Pearce, AnaMaria Perez, Laura Pierson, Emma Wang, Jane Wang, Jing Wang, Rebecca Whitman, Muskan Yadav

Here are three properties of the powers of 2. Can you prove them? (1) For any integer d greater than 1, the sequence of remainders obtained by dividing successive powers of 2 by d is periodic. (2) The sum of the first n powers of 2 (starting with the zeroeth power) is one less than the $(n + 1)$-st power of 2. (3) If you take any string of digits that begins with a digit other than zero, there exists a power of 2 that begins with that string of digits.

Session 31 - Meet 6 October 13, 2022 | Mentors: Elisabeth Bullock, Cecilia Esterman, Jenny Kaufmann, Tharini Padmagirisan, Kate Pearce, AnaMaria Perez, Laura Pierson, Jane Wang, Jing Wang, Rebecca Whitman, Muskan Yadav

Some members tried to figure out how a card trick works. There are many card tricks that connect neatly with mathematics. Can you invent one?

Session 31 - Meet 7 October 20, 2022 | Mentors: Elisabeth Bullock, Jade Buckwalter, Cecilia Esterman, Abhilasha Jain, Jenny Kaufmann, Kate Pearce, AnaMaria Perez, Laura Pierson, Jane Wang, Rebecca Whitman, Muskan Yadav

If you are familiar with single-variable calculus, but have not yet studied multivariable calculus, think about this question: Let $f(x, y) = x^2 + y^2$. It's graph is the surface in 3D given by the points (x, y, z) which satisfy the equation $z = f(x, y)$. At the point $(a, b, a^2 + b^2)$ on its graph, there is a plane tangent to the graph. This plane can be specified by an equation of the form $z = Ax + By + C$. Determine A, B, and C as functions of a and b.

Session 31 - Meet 8 October 27, 2022 | Mentors: Elisabeth Bullock, Jade Buckwalter, Abhilasha Jain, Jenny Kaufmann, Tharini Padmagirisan, Kate Pearce, AnaMaria Perez, Laura Pierson, Rebecca Whitman, Muskan Yadav

If you are given the vertices of a parallelogram in 3D space by specification of their coordinates, can you determine the area of the parallelogram? Note that the parallelogram need not be parallel to any of the coordinate planes. For example, consider the parallelogram whose vertices are (0, 0, 0), (3, 4, 5), (1, 10, 6), and (4, 14, 11). What is its area?

Can you figure out all solutions to the Diophantine equation $a^2 + b^2 + c^2 = n^2$ in integers a, b, c, and n? (This is a generalization of Pythagorean triples to 3D.)

Calendar

Session 31: (all dates in 2022)

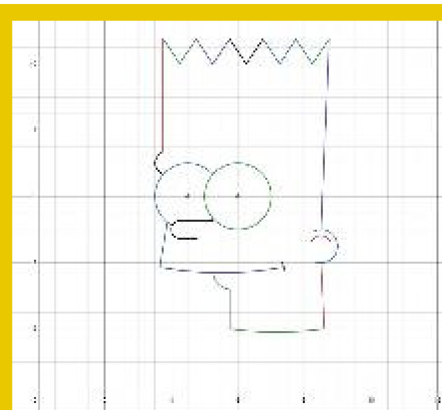
September	8	Start of the thirty-first session!
	15	
	22	
	29	
October	6	
	13	
	20	
	27	
November	3	
	10	
	17	
	24	Thanksgiving - No meet
December	1	

Session 32 (tentative schedule): (all dates in 2023)

January	26	Start of the thirty-second session!
February	2	
	9	
	16	
	23	No meet
March	2	
	9	
	16	
	23	
	30	No meet
April	6	
	13	
	20	No meet
	27	
May	4	

Girls' Angle has run over 150 Math Collaborations. Math Collaborations are fun, fully collaborative, math events that can be adapted to a variety of group sizes and skill levels. We now have versions where all can participate remotely. We have now run four such "all-virtual" Math Collaboration. If interested, contact us at girlsangle@gmail.com. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes for small groups on a wide range of topics. For inquires, email: girlsangle@gmail.com.



Cowabunga! solution. See page 4.

Girls' Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ____________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

____-

The $50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- ☐ Enclosed is a check for $50 for a 1-year Girls' Angle Membership.
- ☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

A Math Club for Girls

Girls' Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is $20/meet for members and $30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle is based in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls’ Angle: Club Enrollment Form

Applicant’s Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you’d like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter’s name in any way. Do we have permission to use your daughter’s image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls’ Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls’ Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: _______________________

(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is \$216 for one session (12 meets)	☐ I will pay on a per meet basis at \$30/meet.
☐ I will pay on a per meet basis at \$20/meet.	☐ I’m including \$50 to become a member, and I have selected an item from the left.

☐ I am making a tax free donation.

Please make check payable to: **Girls’ Angle**. Mail to: Girls’ Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: ___________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____