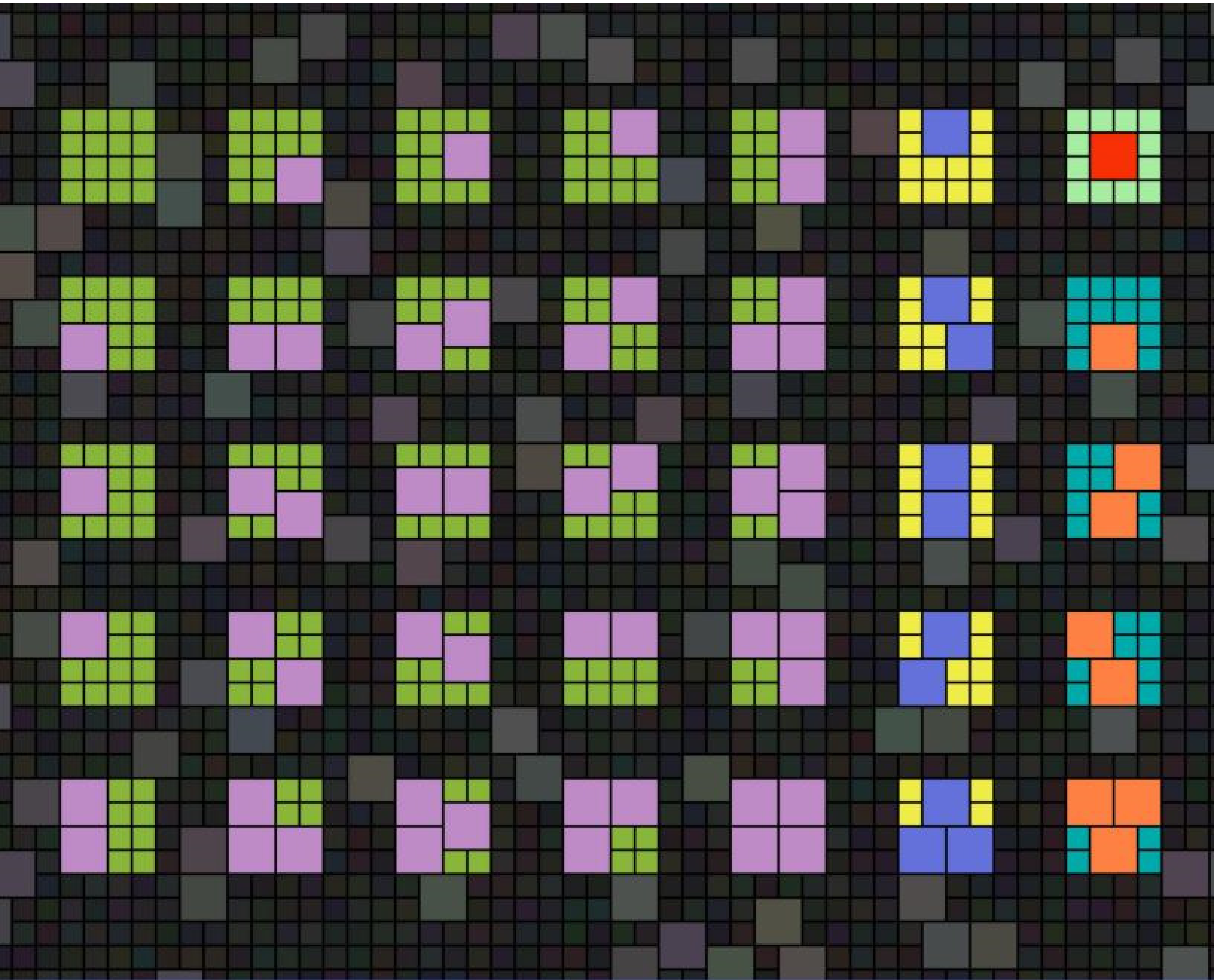
Girls' Angle Bulletin

December 2021/January 2022 • Volume 15 • Number 2

To Foster and Nurture Girls' Interest in Mathematics



An Interview with Tullia Dymarz, Part 2
On the Number of Tilings of a 4-By-*N* Rectangle with 1-By-1 and 2-By-2 Squares
The Needell in the Haystack: When Data Goes Missing

Learn by Doing: Cayley Graphs
All That Math
Notes from the Club

From the Founder

In math, maintaining self-belief in one's problem-solving ability is difficult because there are so many problems that have thwarted all efforts. Learn to modify and simplify questions, and don't give up! As Gian-Carlo Rota said, "Push where the problem yields." - Ken Fan, President and Founder

Girls' Angle Donors

A Heartfelt Thank You to our Donors!

Individuals

Uma Achutha
Nancy Blachman and David desJardins, founders of the Julia Robinson Mathematics Festival, jrmf.org.
Bill Bogstad
Ravi Boppana
Lauren Cipicchio
Merit Cudkowicz
Patricia Davidson
Ingrid Daubechies
Anda Degeratu
Kim Deltano
Concetta Duval
Glenn and Sara Ellison
John Engstrom
Lena Gan
Jacqueline Garrahan
Courtney Gibbons
Shayne Gilbert
Vanessa Gould
Rishi Gupta
Larry Guth
Andrea Hawksley
Scott Hilton
Delia Cheung Hom and Eugene Shih
David Kelly
Mark and Lisel Macenka
Brian and Darline Matthews
Toshia McCabe
Mary O'Keefe
Stephen Knight and Elizabeth Quattrocki Knight
Junyi Li
Alison and Catherine Miller
Beth O'Sullivan
Robert Penny and Elizabeth Tyler
Malcolm Quinn
Jeffrey and Eve Rittenberg
Craig and Sally Savelle
Eugene Shih
Eugene Sorets
Diana Taylor
The Waldman and Romanelli Family
Marion Walter
Andrew Watson and Ritu Thamman
Brandy Wiegers
Brian Wilson and Annette Sassi
Lissa Winstanley
The Zimmerman family
Anonymous

Nonprofit Organizations

Draper Laboratories
The Mathenaeum Foundation
Orlando Math Circle

Corporate Donors

Adobe
Akamai Technologies
Big George Ventures
D. E. Shaw
John Hancock
Maplesoft
Massachusetts Innovation & Technology Exchange (MITX)
Mathenaeum
MathWorks, Inc.
Microsoft
Microsoft Research
Nature America, Inc.
Oracle

For Bulletin Sponsors, please visit girlsangle.org.

Girls' Angle Bulletin

The official magazine of Girls' Angle: A Math Club for girls
Electronic Version (ISSN 2151-5743)

Website: www.girlsangle.org
Email: girlsangle@gmail.com

This magazine is published six times a year by Girls' Angle to communicate with its members and to share ideas and information about mathematics.

Girls' Angle welcomes submissions that pertain to mathematics.

The print version of the Bulletin is printed by the American Mathematical Society.

Editors: Amanda Galtman
Jennifer Sidney Silva
Executive Editor: C. Kenneth Fan

Girls' Angle: A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: *A Telling Tiling* by C. Kenneth Fan. For context, see page 5, in particular, the recursion formula on page 10.

An Interview with Tullia Dymarz, Part 2

In Part 1, Tullia explained the concept of quasi-isometry and defined the lamplighter group.

Ken: What kinds of things are you trying to understand about this [lamplighter] group?

Tullia: The specific thing that I was trying to understand was for a generalization of the lamplighter group to where each lamp can have more than two states. You can imagine, for example, that instead of all the lamps having on or off states, you could have all the lamps have red, green, or yellow states. It was known which of these lamplighter groups were quasi-isometric to each other. But it wasn't known whether there was always a bijective quasi-isometry. I showed that, in fact, some of these that are quasi-isometric, you can't do it through a bijective quasi-isometry.

Ken: Neat! So, you mean the lamplighter groups corresponding to lights that have different numbers of states can sometimes be quasi-isometric…

Tullia: They can be quasi-isometric to each other, but not necessarily in a bijective way.

Ken: Is it known exactly which ones are quasi-isometric, and which ones are not?

Tullia: Yes. Now, it is.

Ken: From your work, do you also know exactly which ones can be shown to be quasi-isometric in a bijective way?

When you're first doing something new and original, you're not going to do it in the nicest way. It's going to be a giant mess that you're going to look at and say, "This is ugly… this is not the right way to do it." But it's only much later that you'll sometimes be able to find a better solution.

Tullia: Yes, yes, yes. And, recently, people have shown even more. So, I told you that the lamplighters are walking along this line, but you can imagine that the lamplighters are walking on a different group, for example, Z squared,[1] or even crazier groups. They were able to determine which of those are quasi-isometric, and which are bijectively quasi-isometric.

Ken: Wow! How many different quasi-isometric classes are there – are there infinitely made from these?

Tullia: Yes, yes. And the existence of a bijective correspondence is determined by the prime factors of n, where n is the number of states the lamps can have.

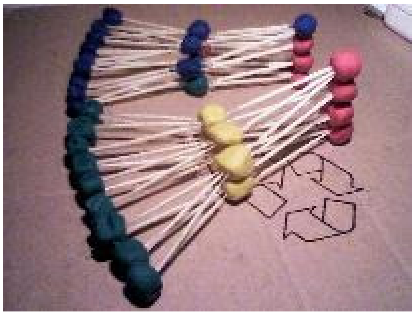
Ken: Really? Wow!

Tullia: Yes. And by the way, it's like I said, this comes back to being able to divide by two. Remember I said that earlier on?

Ken: Yes.

Tullia: Like for the two-state lamplighter group, there's sort of a natural map that

[1] By "Z squared," Tullia means the integer lattice, i.e., the points in the coordinate plane with integer coordinates. So instead of having an infinite row of lamps, you've got an infinite orchard of lamps.



"divides by two." [Tullia picks up the pipe cleaner model and collapses it, bringing two yellow nodes together.] Like here, this has valence two and this has valence two, so I just kind of map them onto each other. But if suddenly you had valence three everywhere, and you had a third one sticking out here, there's no way to divide by two. That was the key observation.

Ken: How did you get this inspiration? How did you come to these realizations?

Tullia: Well, honestly, to really understand, I built these little models. And I saw exactly this, that you could somehow squish these together, but not if you had valence three. This really came out of a question that my thesis advisor had asked me, and it kind of blossomed into this.

Ken: How do you take these finite models, and reason about the infinite from them?

Tullia: In a sense, that's the best we can do, any time we build any models. But what enables us to imagine the infinite comes down to symmetry. It's that these things are very repetitive, so every piece of this infinite graph looks like this [pipe cleaner model]. But now, I can imagine that I have two more. It's the same way that when you draw a grid, you can imagine it going off to infinity. So, it's the symmetry that helps you visualize it.

Ken: Does one's intuition ever go awry, just because it is something that we cannot, actually, have physically?

Tullia: Oh, yes. Sometimes I make assumptions that end up being completely wrong when I actually write them down. But I think with geometry, you need to first try to frame things intuitively, before you start writing it down.

And, actually, I think this is why geometry papers are often very hard to read, because the intuition behind something might be straightforward, but to really set it down on paper requires a lot of extra notation, extra definitions.

Geometry is not like number theory where you are dealing with the numbers. The way math is done now, you have to translate this geometric object onto paper in terms of constants, definitions, and things like that. So, it's a little challenging.

Ken: Yes. I was even thinking of how you would give a rigorous definition for quasi-isometry. It's not clear at all to me how you do that.

Tullia: Yes. And I could do that, but I think it's much more useful to think about it in terms of what you can actually do to a space, like this notion of taking a finite chunk, and squishing it to a point. I can record it precisely using an inequality, but sometimes when you just see the inequality, you don't know that, "Oh, it just means I can take this chunk and squish it to a point."

So, there's actually an extra translation going on when you're doing something like geometry or topology, between the way it's denoted in symbols, versus the picture you have in your head.

Ken: Would you advise a student who's trying to learn about your field and bring life to the symbols, to create and play with such models?

Dear Reader,

We're committed to producing quality math educational content and make every effort to provide this content to you for free.

We are also committed to surviving as a nonprofit!

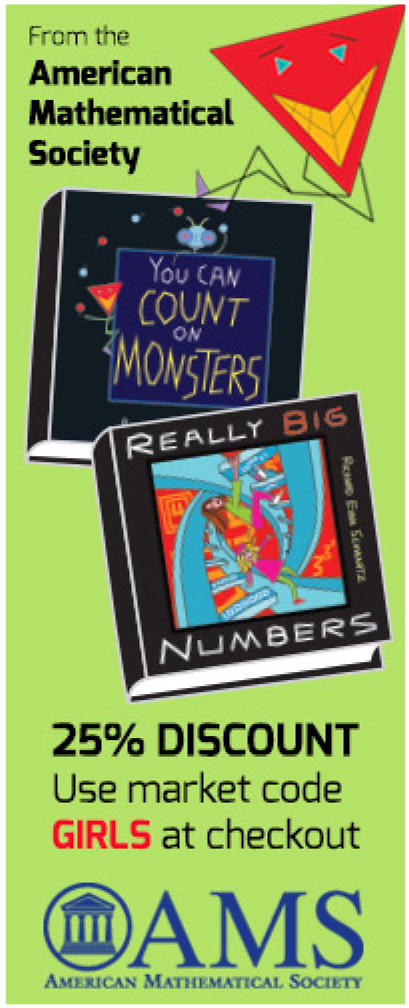
For this issue, those who do not subscribe to the print version will be missing out on a portion of this interview with Prof. Tullia Dymarz and other content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

We know that mathematical interest and talent is unrelated to economic status, which is why we provide so much content for free. But we hope that those of you who are comfortable financially will help us to continue in our efforts.

So, please consider subscribing to the Bulletin. Thanks to our sponsors, subscriptions cost $36/year. With a subscription, you have also gained access to our mentors via email and the ability to influence content in this Bulletin. Visit www.girlsangle.org/page/bulletin_sponsor.html for more information.

Thank you and best wishes,
Ken Fan
President and Founder
Girls' Angle: A Math Club for Girls

Content Removed from Electronic Version



The American Mathematical Society is generously offering a 25% discount on the two book set *Really Big Numbers* and *You Can Count On Monsters* to readers of this Bulletin. To redeem, go to http://www.ams.org/bookstore-getitem/item=MBK-84-90 and use the code "GIRLS" at checkout.

Content Removed from Electronic Version

On the Number of Tilings of a 4-By-*N* Rectangle with 1-By-1 and 2-By-2 Squares

by Diego Abadie, Jonathan Andreoli, Sofia Egan, Tia Reddy,
David Xiong, Yancheng Zhao, and Annie Zhu
edited by Amanda Galtman

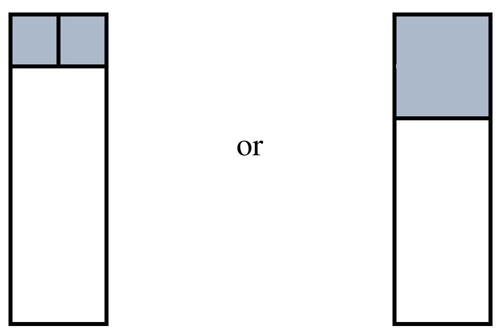
Let S_n be the number of ways to tile a 4-by-n rectangle with 1-by-1 and 2-by-2 squares. In this paper, we give recursive formulas for S_n, two known and one apparently new.

Throughout this paper, whenever we refer to an m-by-n rectangle, we imagine that it is oriented so that the side of length m is horizontal and the side of length n is vertical. Also, we find it convenient to define $S_0 = 1$.

Preliminaries

To present some of our results, we need to know the number of ways to tile a 2-by-n rectangle with 1-by-1 and 2-by-2 tiles. Let f_n be the number of tilings of a 2-by-n rectangle using 1-by-1 and 2-by-2 tiles. It is well known that $f_n = f_{n-1} + f_{n-2}$, for all $n > 2$, and that $f_1 = 1$ and $f_2 = 2$.

To see this, note that when filling the 2-by-n rectangle, for $n > 2$, there are two options for what can appear in the top row: two side-by-side 1-by-1 squares that fill up just the top row, or a 2-by-2 square that fills the top two rows.



We may therefore partition the set of tilings of a 2-by-n rectangle into two sets, one consisting of those with two 1-by-1 tiles side by side in the top row, and the other consisting of those with a 2-by-2 tile occupying the top two rows. The number of tilings in the former set is f_{n-1} and the number of tilings in the latter set is f_{n-2}. Since every tiling is in one of the two sets and no tiling is in both sets, we see that $f_n = f_{n-1} + f_{n-2}$.

When $n = 1$, there is no room for a 2-by-2 tile, so the only way to tile a 2-by-1 rectangle is to place two 1-by-1 tiles side by side. Hence, $f_1 = 1$.

When $n = 2$, we can place a single 2-by-2 tile or fill the 2-by-2 rectangle with four 1-by-1 tiles. Hence, $f_2 = 2$.

We find it convenient to define $f_0 = 1$.

We will make use of the following lemma:

Lemma 1. Any 2-by-2 tile contained in the middle two columns of a 4-by-n rectangle must be flanked by four 1-by-1 tiles.

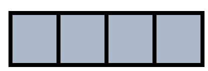
Proof. Suppose that there is a 2-by-2 tile contained in the middle two columns of a 4-by-n rectangle. Because the space to the tile's left and right are only 1 unit wide, both spaces must be filled with two 1-by-1 tiles. □

Note that Lemma 1 implies that any 2-by-2 tile contained in the middle two columns of a 4-by-n rectangle sits in a 4-by-2 subtiling consisting of the two rows that contain the 2-by-2 tile.

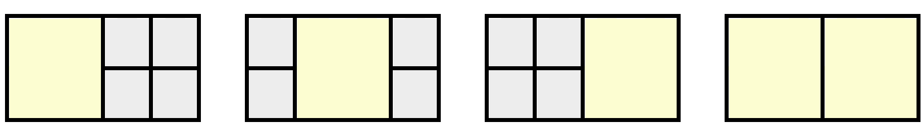
Irreducible Blocks

We call a tiling of a 4-by-n rectangle with 1-by-1 and 2-by-2 tiles an **irreducible block** if and only if it cannot be split by a horizontal line into 4-by-x and 4-by-y subtilings with x, $y < n$. Heubach [2] refers to these as "basic blocks." Heubach showed that there is one 4-by-1 irreducible block, four 4-by-2 irreducible blocks, and two 4-by-n irreducible blocks for $n > 2$. We provide a proof of this here.

Any 4-by-1 tiling is an irreducible block, and there is only one 4-by-1 tiling, so there is one irreducible block of size 4-by-1.



The 4-by-2 tiling with no 2-by-2 tiles can be split in half horizontally and is not irreducible. So any 4-by-2 tiling that is irreducible must have a 2-by-2 tile, and since a 2-by-2 tile cannot be split, any 4-by-2 tiling with a 2-by-2 tile is an irreducible block. The reader may verify that there are exactly four such tilings:

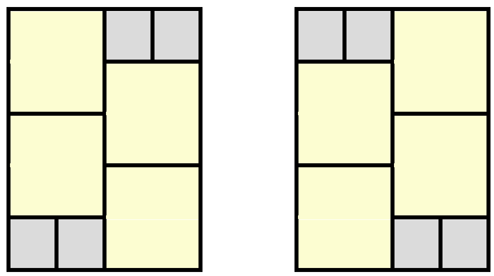


For $n > 2$, we show that there are exactly two irreducible blocks of size 4 by n. By Lemma 1, if the middle columns have a 2-by-2 tile, there must be only 1-by-1 tiles flanking it. This means that irreducible blocks cannot have a 2-by-2 tile in the middle columns of the rectangle.

In the upper left corner, there must be a 1-by-1 tile or a 2-by-2 tile. Suppose we put a 1-by-1 tile in the corner. Because of the above observation, we must also put a 1-by-1 tile on its right; without that, we would have to put a 2-by-2 tile in the middle two columns. After putting a 1-by-1 tile next to the first 1-by-1 tile, we must put a 2-by-2 tile to the right of both 1-by-1 tiles, because otherwise we would have to place two 1-by-1 tiles next to the first two in the first row, which would result in a non-irreducible block. After putting the 2-by-2 tile down, we are forced to put down another 2-by-2 tile just below the first two 1-by-1 tiles, because otherwise we would have to complete the top two rows with two more 1-by-1 tiles and we would have a non-irreducible block. This pattern continues until the last row, which has two 1-by-1 tiles beside each other next to a 2-by-2 tile.

Alternatively, suppose we put a 2-by-2 tile in the upper left corner. Then we must put two 1-by-1 tiles next to this tile, since placing another 2-by-2 tile next to it would result in a non-irreducible block. By reasoning similar to that in the prior paragraph, the remaining tiles are forced, and we obtain a unique irreducible block that happens to be the mirror image of the irreducible block found in the previous paragraph.

We conclude that there are exactly two irreducible blocks of size 4-by-n, when $n > 2$.



The two irreducible 4-by-5 blocks.

Recursive Formula 1

Our first recursive formula for S_n is:

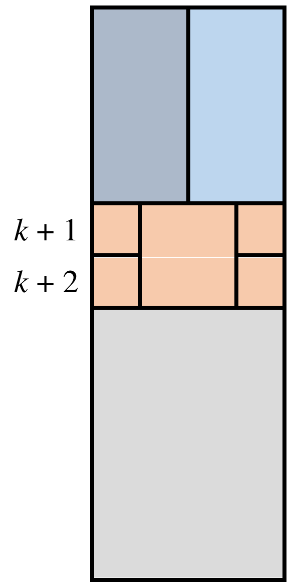
$$S_n = f_n^2 + \sum_{k=0}^{n-2} f_k^2 S_{n-k-2} \,,$$

for $n \geq 2$. This formula is based on partitioning the tilings based on the location of the highest 2-by-2 square (if any) appearing in the middle two columns. The first term corresponds to there being no 2-by-2 square in the middle two columns, and the kth term in the summation corresponds to the highest 2-by-2 square in the middle two columns occupying rows $k + 1$ and $k + 2$.

First, if there is no 2-by-2 tile in the middle two columns, we can split the rectangle down the middle to form two 2-by-n tilings. Since any choice of tiling of a 2-by-n rectangle may be used for either half, there are f_n^2 such tilings.

To find the rest of the possibilities, we partition the remaining tilings according to the location of the topmost 2-by-2 square in the middle two columns, which we refer to as Q. Let k be the row just above Q. (If Q is at the top of the rectangle, we take $k = 0$.)

The number of such tilings is calculated by breaking the 4-by-n rectangle into four sections: the two rows that contain Q, the 4-by-$(n - k - 2)$ section below Q, and the two 2-by-k rectangles obtained by splitting the section of the rectangle above Q vertically down its middle. (We can split the section above Q in this way because there is no 2-by-2 tile in its middle two columns).



Sectioning a 4-by-n tiling whose topmost 2-by-2 tile in the middle two columns occupies rows $k + 1$ and $k + 2$.

To build a tiling of a 4-by-n rectangle with a 2-by-2 tile occupying the middle two columns of rows $k + 1$ and $k + 2$, we are free to specify any tiling of each of the sections described above except for the section containing Q. There are S_{n-k-2} possibilities for the second section described, and there are f_k possibilities for each of the remaining sections. This gives us a total of $f_k^2 S_{n-k-2}$ such tilings.

If we add these up over all possible values of k, we obtain our first recursive formula.

Recursive Formula 2

A second recursive formula for S_n, which is due to Heubach [2], is:

$$S_n = S_{n-1} + 4S_{n-2} + 2(\sum_{k=0}^{n-3} S_k) .$$

To explain this, observe that we can partition the set of tilings of a 4-by-n rectangle with 1-by-1 and 2-by-2 tiles according to how far down you must go to find the first horizontal line that can be drawn that splits the tiling into two subtilings. If the line occurs between row j and $j + 1$, then above the line, there must be an irreducible block of size 4 by j. Below the line, there can be any tiling of a 4-by-$(n - j)$ rectangle.

Since any 4-by-j irreducible block may be used with any tiling of a 4-by-$(n - j)$ rectangle, the number of such tilings is $b_j S_{n-j}$, where b_j is the number of irreducible blocks of size 4 by j.

Summing over all possible values of j, we obtain the second recursive formula.

Recursive Formula 3

The third recursive formula for S_n that we present is mentioned by Schneider [1] in the On-Line Encyclopedia of Integer Sequences. It is listed in entry A054854 without proof or a reference. It is:

$$S_n = 2S_{n-1} + 3S_{n-2} - 2S_{n-3}.$$

We provide a proof here. The proof uses our second recursive formula for S_n.

Proof. Our second recursive formula is

$$S_n = S_{n-1} + 4S_{n-2} + 2(\sum_{k=0}^{n-3} S_k).$$

By adding and subtracting $2S_{n-2} + 2S_{n-1}$, we can rewrite this as

$$S_n = 2S_{n-2} - S_{n-1} + 2(\sum_{k=0}^{n-1} S_k).$$

Therefore,

$$\begin{aligned}
S_n &= 2S_{n-2} - S_{n-1} + 2(\sum_{k=0}^{n-1} S_k) \\
&= 2S_{n-2} - (2S_{n-3} - S_{n-2} + 2\sum_{k=0}^{n-2} S_k) + 2(\sum_{k=0}^{n-1} S_k) \\
&= 2S_{n-2} - 2S_{n-3} + S_{n-2} - 2\sum_{k=0}^{n-2} S_k + 2(\sum_{k=0}^{n-1} S_k) \\
&= 2S_{n-2} - 2S_{n-3} + S_{n-2} + 2S_{n-1} \\
&= 2S_{n-1} + 3S_{n-2} - 2S_{n-3}
\end{aligned}$$

which is Schneider's recursive formula. □

Bibliography

[1] OEIS Foundation Inc. (2021), Entry A054854 in The On-Line Encyclopedia of Integer Sequences, https://oeis.org/A054854

[2] S. Heubach, Tiling an m-by-n Area with Squares of Size up to k-by-k ($m \leq 5$), *Congressus Numerantium* 140 (1999), 43-64.

Editor's note: All authors are students at the Buckingham, Browne, and Nichols Middle School in Cambridge Massachusetts.

The Needell in the Haystack[1]

When Data Goes Missing

by Anna Ma | edited by Jennifer Sidney Silva

More often than not – especially when working with real-world data – we have to work with messy, noisy, inaccurate, and *incomplete* data. Otherwise, we might incessantly get emails that look like this:

Dear Girls' Angle Bulletin *Reader,*

This is your favorite streaming service provider and we noticed that you have not finished watching and rating all 3500 of the shows available on our platform. In order for us to run a regression model to predict your movie preferences, we need you to complete all 3500 of your ratings as soon as possible, so please watch and submit your ratings at your earliest convenience.

Yours truly,
Streaming Company X

Figure 1. Example 7×4 matrix of user ratings for different movies with missing entries (black blocks).

Luckily, we don't come across such emails because there are ways that we can deal with incomplete data like the data shown in Figure 1, where 7 users' ratings are organized into a matrix A. In this article, we will discuss how we can go about solving large-scale linear systems even if our matrix A has missing entries.

In the above example, data is missing because practically speaking, users cannot watch every single movie or show in a streaming company's database and rate it. This is just one reason data can be missing from a data set. In some settings, the data is simply too expensive for a company or group to acquire, so they have to limit the amount of data collected. For example, while in-person interviews tend to give social scientists the most detailed and reliable data, it is the most time-consuming and expensive way to acquire data. An interviewer may choose to skip questions in order to save time, causing portions of data for that interviewee to be missing. It is important to keep in mind how and why an interviewer chose to skip a particular question; this will be discussed further in the next section. As another example, missingness may stem from how one goes about data collection. For example, if interviews are voluntary, we may find ourselves with data missing from different groups. Perhaps children are more likely to volunteer to discuss their day than adults are. Human and technical error can also play a part in the lack (or corruption) of data. An interviewer may simply forget to ask a couple of questions on the last page of the interview.

---

[1] This content is supported in part by a grant from MathWorks. Anna Ma is a Visiting Assistant Professor at the University of California, Irvine.

While these are all sources and examples of why data can be missing from our dataset, they cannot be treated equally.

Types of Missing Data

Before addressing how to deal with missing data, we must first identify the *type* of missing data we face, which will impact how we treat it. It should be noted that the way we classify the missing data will depend on the *assumptions* we make about how and why the data is missing. There are three main types of missing data: **missing completely at random**, **missing at random**, and **missing not at random**.

Figure 2 provides a visualization of these three types of missing data. In each of the three subfigures, we have a 10×10 matrix indicated by the 10×10 blocks. The blocks that are colored in black are missing from the data matrix, and the blocks that are colored in white are not missing. In the left subplot, the missingness seems to have absolutely no structure or predictable behavior. This is an example of what we expect to see when data is missing completely at random. In the middle subfigure, we have structured missingness. Here, the rows of the matrix have been reordered so that one particular subset of the data appears in the first half of the matrix. This is what we expect to see when data is missing at random. The randomness in this case is whether the row belongs to the top or bottom half of the data set after reordering. Lastly, in the right subfigure, we see that we have entries missing. Compared to the first two subfigures, it looks more like the second figure where there are longer, consecutive blocks of missing data; but here we assume that there is no way to sort the data to explain why certain blocks are missing. In this case, this data set would be considered missing not at random.

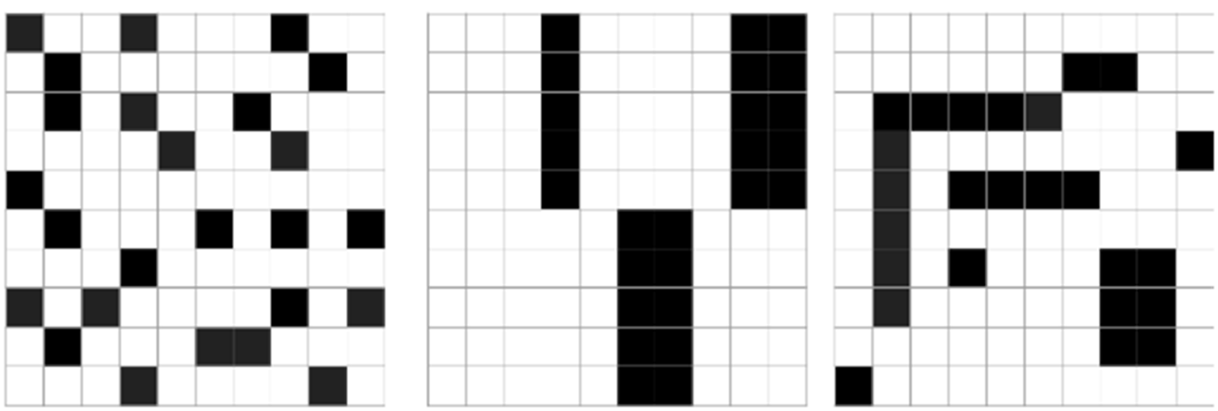


Figure 2. Visualization of different types of missingness in 10×10 matrices. (Left) Missing completely at random (Center) Missing at random (Right) Missing not at random.

When the likelihood that a data point is missing is the same across all data points in a data set, we say this data is missing completely at random (MCAR). The main disadvantage of this type of missingness is that it is rarely found naturally in real-world data sets. However, it can still be quite useful. For example, to save on memory storage, instead of storing a large $n \times n$ matrix with low complexity, or *rank* r (this would cost on the order of n^2 bits to store), you can instead store a random subsample of the matrix (much less than the n^2 bits) and recreate your matrix using matrix completion algorithms. Another example of when MCAR can be leveraged is when we want to avoid survey fatigue. Survey fatigue is a phenomenon that occurs when people take long surveys and either stop taking the survey altogether or start answering questions carelessly. To minimize survey fatigue, we may choose to randomly select 10 of the 30 questions to ask survey takers instead of asking each one all 30 questions. While this creates missing information in our data set, if we are truly randomly selecting which questions we query survey takers, strategies for addressing MCAR data can be leveraged to account for it.

When the likelihood that a data point is missing is conditional on some observed information, the data set is said to be missing at random (MAR). For example, an interviewer would not ask a male interviewee, “Have you ever been pregnant?” *Conditioned* on the gender of an interviewee, the chances of being asked the question is always the same: given that an interviewee’s gender is male, he has a 0% chance of being asked this question; and if the interviewee’s gender is female, she has a 100% chance of being asked this question. Note that here, the missing value is dependent on *observed* data (we observed the gender). The advantage of this type of missing data is that we can typically draw inferences and complete the missing data using observed information. It is also more common in real-world applications than its stricter MCAR counterpart. However, it is still not as common as the last type of missingness: missing not at random.

If missingness of a data set does not fall into the above two types, that data is said to be missing not at random (MNAR). This typically insinuates that there is a systematic, *unobserved* reason that data is missing, and it is the most difficult but most common type of missingness we encounter in real-world data. For example, suppose that in a forest we have 10 sensors measuring anything from humidity to temperature in 1-hour increments. One of the sensors breaks, and we now have missing data from that sensor for the next 24 hours (supposing that’s how long it takes a technician to come out and fix the sensor). The missingness in this data set is systematic – the likelihood that a data point is missing is dependent on whether the sensor was working and if it wasn’t working, how long it had been broken. While this type of missingness tends to be the most commonly encountered type of missingness, it’s the most complex and discipline dependent.

Ways to Handle Missing Data

Figure 3. Deletion in action!

There are three main ways to deal with missing data: **deletion**, **imputation**, and **analysis**. In the deletion approach, we are choosing to *ignore* the fact that we have missingness in our data set and just remove all rows and columns of our matrix that have *any* missing entry. This means that if just one entry of your row is missing, you will choose to remove that entire row. Potentially, that is a lot of data we are throwing away! In fact, if we look back at Figure 2, not a single row or column would be kept in the first two subfigures, and the last subfigure on the right would only have one row remaining (the first). Clearly, deletion is not a great choice when there is a significant amount of data missing.

Imputation is another way to address missing data. In imputation, we *impute* (input and compute) the missing values with a dummy value, such as 0 or -1, or a statistic, such as the mean or median value. It’s important to note that imputation only works when data is MCAR or MAR. If data is MNAR, imputation may cause bias in the completed data. In the forest sensor example of MNAR above, if abnormally cold temperatures caused the sensor to malfunction and stop working, imputing the average temperature in the missing data would skew the data set and make it biased towards the average, when our data should, in fact, represent an abnormality in temperature. You can also imagine that imputing the average temperature would be impractical in this case; you would expect to see temperature that slowly increases and decreases over time, but we would see a drastic change in temperature due to the imputation.

Analysis is a more sophisticated way of dealing with missing data; it encompasses anything outside of the standard deletion and imputation approaches. In the forest sensor example, instead of replacing missing values with the mean, we could interpolate between the last two temperatures measured to ensure a smooth transition between the last measurement we had before the sensor broke and the first measurement we had after the sensor was fixed. As another example, we can use optimization algorithms to complete matrices under certain constraints (such as how complex the matrix is). This large area of mathematical research falls under the aptly named **Matrix Completion Problem**. As a last example, if we want to use a data set with missing data to solve a regression problem, we could adapt our algorithm to the fact that we have missing data. We will demonstrate this using the Stochastic Gradient Descent algorithm.

Stochastic Gradient Descent when Data Goes Missing

In our last installment of "Needell in the Haystack," we discussed solving large-scale linear systems using an algorithm called **Stochastic Gradient Descent (SGD)**. As a refresher, consider solving the linear system $Ax = y$ via the minimization of the **Least Squares Objective**: $F(x) = \|Ax - y\|^2$, where A is an $m \times n$ matrix, x is an unknown signal, y is the given m-dimensional measurement vector, and $\|\cdot\|^2$ denotes the squared Euclidean norm.

SGD is an algorithm which iteratively approximates the gradient of $F(x)$ in order to update an estimated solution to the linear system $Ax = y$. In particular, when $F(x)$ can be written as a sum of some component functions, i.e., when we can write $F(x) = \sum f_i(x)$, the iterations of SGD are written as $x_{t+1} = x_t - \alpha \nabla f_i(x_t)$, where f_i is a randomly chosen component. If $\nabla f_i(x_t)$ are **unbiased** approximations of $\nabla F(x_t)$ (that is, the expected value of $\nabla f_i(x_t)$ is $\nabla F(x_t)$), then on average, we will be moving in the direction of the full gradient. This is exactly the reason why SGD works so well for this problem: we have unbiased approximations of $\nabla F(x)$. However, if we have missing data, these approximations may not be unbiased at all! So, we need to change our update function to adapt to the fact that we have missing data.

In the remainder, the computations are fairly extensive, and we omit many details. We urge readers who are comfortable with linear algebra and probability to fill in missing details.

To that end, let's call our new update function $g_i(x)$. Our goal is to get $g_i(x)$ to be an unbiased estimator of $\nabla F(x)$. In order to compute expectations, we need to make assumptions about the type of missing data in matrix A. For simplicity, we start with the MCAR data. To make this concise, we assume the following: each entry of A has a probability p of not being missing. Let's suppose we don't have access to the entire completed matrix A, but instead we have a matrix $\tilde{A}$ with missing data. We relate $\tilde{A}$ to A by writing $\tilde{A} = A \circ D$, where D is an $m \times n$ random **Bernoulli** matrix of 0s and 1s and $\circ$ denotes a Hadamard or **element-wise** product. We can think of D as an indicator matrix: if an element of the matrix is 1, it indicates that the element is not missing. If an element of D is 0, then the corresponding data value is missing. That means that $D_{ij} = 1$ with probability p and $D_{ij} = 0$ with probability $1 - p$. Now, $\tilde{A}$ is the matrix we have access to and the least squares objective that we are minimizing is $\tilde{F}(x) = \frac{1}{2}\left\|\tilde{A}x - y\right\|^2$, which we can write as $\tilde{F}(x) = \frac{1}{2}\left\|\tilde{A}x - y\right\|^2 = \frac{1}{2}\sum_{i=1}^{m}(\tilde{A}_i x - y_i)^2 = \sum_{i=1}^{m}\tilde{f}_i(x)$, where $\tilde{f}_i(x) = \frac{1}{2}(\tilde{A}_i x - y_i)^2$, $\tilde{A}_i$ is row i of matrix $\tilde{A}$, and y_i is the ith component of vector y. If we apply SGD naively, we would have $\nabla\tilde{f}_i(x) = \tilde{A}_i^T(\tilde{A}_i x - y_i)$. Taking the expectation with respect to the randomly selected row and missing data model, we have that

$$E[\nabla\tilde{f}_i(x)] = \frac{1}{m}(p^2 A^T(Ax - y) + (p - p^2)diag(A^T A)x).$$

Thus, $\nabla\tilde{f}_i(x)$ is not an unbiased estimator for $\nabla F(x) = A^T(Ax - y)$ and we cannot let $g_i(x)$ be $\tilde{f}_i(x) = \frac{1}{2}(\tilde{A}_i x - y_i)^2$. Our goal is to find another function $g_i(x)$ such that $E[g_i(x)] = \frac{1}{m}\nabla F(x)$.

Before we jump into the right choice of $g_i(x)$, let's start with a couple of simpler computations. We first consider the expected value of an entry of $\tilde{A}$ with respect to randomness from the missing data. It is

$$E[\tilde{A}_{ij}] = E[D_{ij}A_{ij}] = A_{ij}E[D_{ij}] = pA_{ij}.$$

Next, we consider fixed vectors x and y. The expected value of $\tilde{A}_i x - py_i$ is

$$E[\tilde{A}_i x - py_i] = \frac{1}{m}p(Ax - y).$$

Lastly, for the expectation of $\tilde{A}_i^T\tilde{A}_i$ and $diag(\tilde{A}_i^T\tilde{A}_i)$ we have

$$E[\tilde{A}_i^T\tilde{A}_i] = \frac{1}{m}(p^2 A^T A + (p - p^2)diag(A^T A))$$

and

$$E[diag(\tilde{A}_i^T\tilde{A}_i)] = \frac{1}{m}(p\ diag(A^T A)).$$

Finally, we see that if we want a function $g_i(x)$ such that $E[g_i(x)] = \frac{1}{m}A^T(Ax - y)$, we should pick

$$g_i(x) = \frac{1}{p^2}(\tilde{A}_i^T(\tilde{A}_i x - y) - (1 - p)diag(\tilde{A}_i^T\tilde{A}_i)x),$$

since

$$
\begin{aligned}
E[g_i(x)] &= \frac{1}{p^2}(E[\tilde{A}_i^T(\tilde{A}_i x - y)] - (1-p)E[diag(\tilde{A}_i^T\tilde{A}_i)]x) \\
&= \frac{1}{m}A^T(Ax-y) + \frac{1}{m}(p-p^2)diag(A^TA)x - \frac{1}{m}(p-p^2)diag(A^TA)x \\
&= \frac{1}{m}A^T(Ax-y) \\
&= \frac{1}{m}\nabla F(x).
\end{aligned}
$$

Using the proposed $g_i(x)$, we have a new iterative method that approximately solves linear systems with missing data: $x_{t+1} = x_t - \alpha_t g_i(x_t)$. The question now is how well does this method work? Does its efficacy scale with the probability that a data point is missing? Intuitively, it should; if $p = 1$, then we should be performing SGD on a data set without missing data. Can you hypothesize how well this algorithm will work with respect to the probability data is available p? To see if your hypothesis is correct, we can investigate empirical numerical results using this algorithm. Figure 4 shows these experiments where the probability of data availability is varied. Note that when $p = 1$, the algorithm works just as SGD does and converges exponentially. When p is smaller than 1 and we have data missing, the algorithm experiences a *convergence horizon* and can only reach a certain error bound. This is expected, as we don't have *all the data*. Despite this, we can still make a good approximation whose accuracy depends on how much data we have.

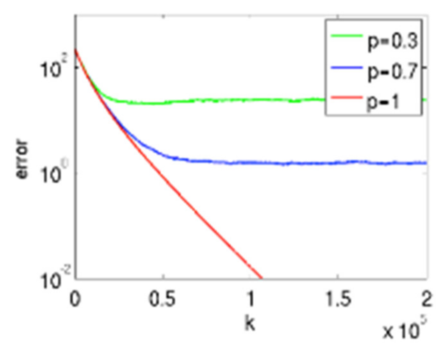


Figure 4. Numerical experiments for different probabilities of missing data.

Here we only analyzed data sets of data MCAR. Whether this approach can accurately extend to MAR and MNAR really depends on how well we can model the missing data. These are still open questions in research, but perhaps you will be the one to develop a good model for MAR or MNAR data and will design your own algorithm for missing data!

References

[1] R. Little and D. Rubin, *Statistical analysis with missing data*. Vol. 793. John Wiley & Sons, 2019.

[2] A. Ma and D. Needell, Adapted stochastic gradient descent for linear systems with missing data, *Numerical Mathematics: Theory, Methods and Applications* (2017).

[3] L. Nguyen, J. Kim, and B. Shim, Low-rank matrix completion: a contemporary survey, *IEEE Access* **7** (2019), 94215-94237.

[4] W. Lin and C. Tsai, Missing value imputation: a review and analysis of the literature (2006–2017), *Artificial Intelligence Review* **53.2** (2020), 1487-1509.

Learn by Doing

Cayley Graphs

by Addie Summer | edited by Jennifer Sidney Silva

In her interview, Prof. Dymarz told us about the lamplighter group and showed us part of her pipe cleaner model of it. The group is a wondrous object to explore, and if you're feeling left out because you're not familiar with Cayley graphs, this "Learn by Doing" is meant for you!

Cayley graphs are a way to visualize structure in an algebraic object known as a **group**. Let's begin by discussing groups.

Groups

We'll start with an example of a group: the symmetries of a square. Here, what we mean by a symmetry of the square is anything you can do to the square that looks like you've done nothing. (Actions that have the same net effect on the locations of the vertices are considered the same action.) For example, you can rotate the square by 90° clockwise about its center. If a person didn't watch you rotate it, they wouldn't think you had moved the square at all.

1. How many symmetries does a square have? Include the symmetry where you do nothing to it.

(Spoiler Alert!) There are 8 symmetries of a square: Do nothing, rotate clockwise by 90°, rotate clockwise by 180°, rotate clockwise by 270°, and flip over any one of its 4 lines of mirror symmetry. (Here, all rotations are taken about the center of the square.) Notice that there are multiple ways of describing the same symmetry. For example, rotating by 90° clockwise is the same as rotating by 270° counterclockwise.

If you have two symmetries of a square, which we'll denote by A and B, you can perform A, followed by B, and the result will be some symmetry C. We can use BA to denote the symmetry obtained by performing A followed by B (yes, the order the symmetries are applied is customarily read from right to left!), borrowing the notation used to denote multiplication. However, unlike with multiplication, order can make a difference. That is, AB is not necessarily the same as BA. For this reason, the ability to combine symmetries A and B to get BA is usually referred to as the **group law** or **group operation**; sometimes people also refer to the group law as **group multiplication**, especially when AB does equal BA for all A and B (in which case, the group is said to be **commutative**, or **Abelian**).

2. Find an example of 2 symmetries of the square, A and B, where $AB \neq BA$.

3. For whole numbers and multiplication, typically people make 12 by 12 multiplication tables, whereas a complete multiplication table would be infinitely large! However, since there are only 8 symmetries of a square, you can make an analog of the multiplication table that shows the result of the group law for *every* pair of symmetries A and B. Make one!

4. Any object has a set of symmetries which can be combined analogously to the way we combined symmetries of the square. Make a table showing the group laws for the set of symmetries of a cube, regular hexagon, or whatever shape you wish.

A group generalizes the concept of the set of symmetries of an object. A group is a set together with a group law, which is a way of combining two elements of the set to obtain a third. The group law must satisfy three properties:

- The group law must be **associative**; that is, for any three elements A, B, and C in the set, it must be true that $(AB)C = A(BC)$.
- There must be an element of the set, which we'll denote by 1, which satisfies $A1 = 1A = A$ for any element A in the set. The element 1 is called the **identity element**.
- For any element A in the set, there must be an element in the set, which we'll denote by A^{-1}, that satisfies $AA^{-1} = A^{-1}A = 1$. The element A^{-1} is called the **inverse** of A.

5. Check that the operation you were using for combining symmetries of an object satisfies all 3 of these properties.

Here are more examples of groups. Play around with each group. Come up with a way to denote its elements and develop facility applying the group law.

6. Let L be a straight line and let R be the set of symmetries of L, where a symmetry is any translation (shifting) of the line left or right by a fixed amount. The resulting group of symmetries is an example of an infinite group because it has infinitely many elements.

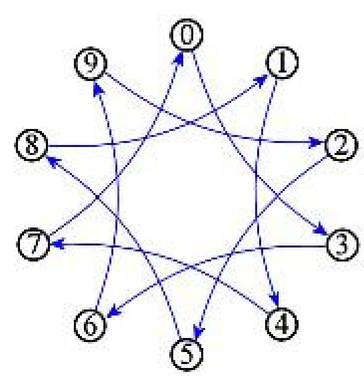
7. Let C_n consist of n dots arranged around a circle. Let Z_n be the set of symmetries of C_n, where a symmetry is restricted to one of the n rotations that shift all the dots by some fixed amount around the circle. This group is called the **cyclic group of size *n***. It is the same group you get with addition, modulo n. (That is, you take the integers, with addition as the group operation, but you consider two integers to be the same if they differ by a multiple of n.)

8. Let D_n be n distinct dots. Let S_n be the set of symmetries of D_n, where a symmetry is any permutation of the dots. Show that there are n! elements in S_n. This group is called the **symmetric group on *n* objects**.

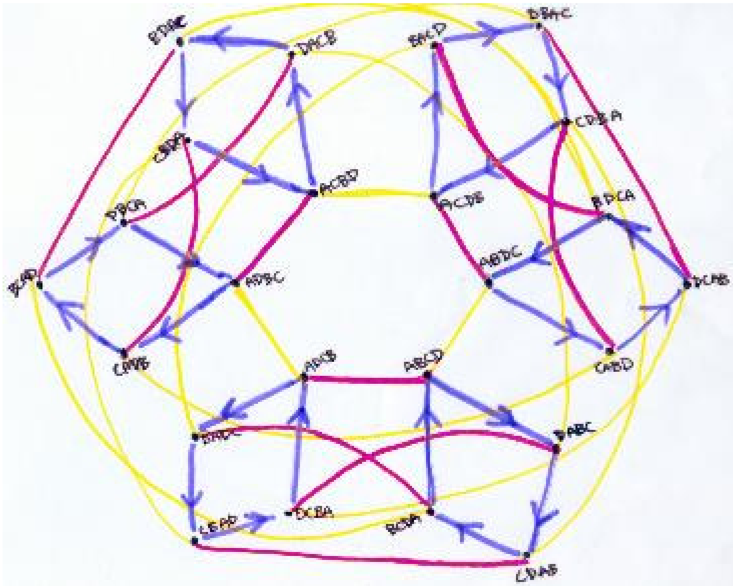
Cayley Graphs

Cayley graphs provide a way to visualize the group operation for specific elements of the group. Let G be any group. To make a Cayley graph, we pick a subset S of G. We then draw a dot (or some symbol) for each element of the group. Next, we draw arrows from one element A to another element B if $B = XA$ for some element X in S.

The figure at right shows an example of a Cayley graph. The group is C_{10} (see Problem 7), and S consists of the single group element corresponding to shifting the 10 dots around the circle 3 dots over clockwise. We've labeled the elements of the group by a number which tells how many dots over the 10 dots are shifted clockwise. For example, the element labeled "0" is the identity element of C_{10}.



9. Verify that this is the Cayley graph as described.



If both X and X^{-1} are in S, then if $B = XA$, we can dispense with the arrow and simply draw an edge from A to B since if $B = XA$, then $A = X^{-1}B$. At left is an example using S_4 created by **404 Name Not Found**,[2] which was originally printed on page 14 in Volume 8, Number 4. The author labeled the elements of S_4 (see Problem 8) by indicating a permutation of the first 4 letters of the alphabet, ABCD. For example, "ACBD" means to swap positions 2 and 3, but to leave positions 1 and 4 in place. Thus, "ABCD" corresponds to the identity element.

10. What set S did **404 Name Not Found** use to create the above Cayley graph of S_4?

11. **404 Name Not Found**'s Cayley graph contains an 8-element red/blue connected subgraph which includes the element "ABCD." Do you recognize these 8 elements as being the same as the group of symmetries of a square?

For more images of Cayley graphs, check out the "Math Buffet" on pages 12-15 of Volume 8, Number 4 of this *Bulletin*, as well as that issue's cover graphic.

12. Pick at least two of your favorite groups and experiment with creating different Cayley graphs of each using different sets S. What does it mean if the resulting Cayley graph is not connected? (Connected means that there is a path joining any two elements of the group in the Cayley graph.)

13. Which groups have the property that they have a connected Cayley graph for some S consisting of only a single element?

Lamplighter Groups

If you're feeling ready to reconstruct Prof. Dymarz's pipe cleaner model of a Cayley graph of the lamplighter group, please go for it!

If not, let's work on a warm-up exercise. One challenge that the lamplighter group presents is that it has infinitely many elements. So, to warm up, we can modify the lamplighter group by using a circular arrangement of lamps, as opposed to the infinite row of lamps lining an infinitely long street.

In Part 1 of her interview, which appeared in the previous issue, Prof. Dymarz introduced the lamplighter group in terms of generators and relations. We imagine an infinite row of streetlamps, and a lamplighter standing before a lamp. All the lamps are turned off. One element of the lamplighter group, call it t, corresponds to the lamplighter toggling the switch of the lamp she is at. Another element of the lamplighter group, call it x, corresponds to the lamplighter moving over one lamp to the right. (Its inverse, x^{-1}, is to move one lamp over to the left.) Every element of the lamplighter group can be written as a word in t, x, and x^{-1}. If two words result in the exact same lamps being turned on *and* the lamplighter standing before the same lamp, the elements of the group represented by those words are considered to be equal. Prof. Dymarz's pipe cleaner model is a model of the Cayley graph of the lamplighter group with respect to the set $S = \{x, x^{-1}, tx, x^{-1}t\}$.

[2] For minors, we sometimes use pseudonyms, which appear in boldface in the Bulletin.

Girls!

Learn Mathematics!

Make new Friends!

Meet Professional Women who use math in their work!

Improve how you Think and Dream!

Girls' Angle

A math club for ALL girls, grades 5-12.

girlsangle@gmail.com
girlsangle.org

Girls' Angle

All That Math

by Lightning Factorial | edited by Jennifer Sidney Silva

To get good at math, do math! Doing math is not only rewarding and beneficial to your own development, but it can also be a lot of fun.

To do math, be curious and think. Inevitably, a question will occur to you that can be thought about rationally and abstractly. Try to answer that question, and you'll be doing math.

We try to illustrate this process frequently in the pages of this magazine, such as with Emily and Jasmine's mathematical explorations. However, one difficulty in writing about the process is that it becomes necessary to illustrate with a specific mathematical example, and the particular example may not interest you. Also, the illustration will have to follow some specific development of ideas, and that, too, may not resonate with the way you think. Ideally, your question would emerge out of your interests and the development would proceed in a way that's natural to you.

The hope is that you will not get caught up with the specific example, especially if it doesn't appeal to you, but instead pick up on how the example is developed through a process of asking and answering questions – a process you can apply to what you *do* find interesting.

Once you get going, you'll see that the math starts to radiate from all corners! Here, we follow Lightning Factorial's thinking process, while separately pointing out all of the math that simply oozes out. On the left are Lightning's thoughts. On the right is some of the math that those thoughts can lead to.

It's going to be the year 2022 soon. It seems like every math contest has some problem that uses the current year. I wonder if I can anticipate some of the questions that will be asked that use 2022.	If you have no other mathematical thoughts, it can be fun to just pick a number and see how much you can say about it.
Are there any special properties of the number 2022? One thing that jumps out is that the number is even. Actually, what is its prime factorization?	At the very least, you can figure out the prime factorization of the number. And once you start doing that, it's natural to ask: What is the fastest way to compute a prime factorization?
If I divide by 2, I get 1011. And since the sum of the digits of 1011 is 3, the number 1011 must be divisible by 3, but not by 9. Let's see, 1011 divided by 3 is… 337. Is 337 a prime number?	(Lightning applies "casting out 9s" here.) When you compute a prime factorization, you end up asking whether a given number is a prime number. What's an efficient way to test whether a number is a prime number?
Well, I already know it's not divisible by 2 and 3, and it's not divisible by 5, 7, or 11. How about 13? If I add 13 to 337, I get 350; and if 13 divides 337, it must divide 350, which means 13 would have to divide 35, which it	Thinking about prime numbers is an entryway that quickly leads to the forefront of mathematics. A wonderful lecture on this topic is "The First 50 Million Prime Numbers," by Don Zagier.

does not, so 337 is not divisible by 13. What about 17? If I subtract 17 from 337, I get 320, and if 17 divides 337, it must divide 320, which means it must divide 32, which it doesn't. So, 337 is not divisible by 17, and since $19^2 = 361 > 337$, I don't need to check any additional primes: 337 is a prime number!

Here, Lightning checks 337 for divisibility by each prime up to the square root of 337. Why does it suffice to check only up to the square root? Are there faster methods?

So the prime factorization of 2022 is:

$$2022 = 2 \times 3 \times 337.$$

I don't see anything particularly special about this prime factorization.

The prime factorization of 2022 also shows that 2022 is "square free," meaning that it is not divisible by a perfect square greater than 1. How many divisors does 2022 have? What can you say about the number of divisors of a square-free number?

I wonder what the nearest perfect square is to 2022. I know that $40^2 = 1600$ and $50^2 = 2500$, so the (positive) square root of 2022 is between 40 and 50. Let's see, $45^2 = 2025$… that's really close! It's got to be the nearest perfect square. So

$$2022 = 45^2 - 3.$$

The idea to look for the nearest perfect square to 2022 may have been influenced by last year being 2021, which differs from 45^2 by a perfect square. Specifically, $2021 = 45^2 - 2^2$. Using the identity $a^2 - b^2 = (a + b)(a - b)$, this leads us to the factorization 43×47 for 2021, a fact used, for example, in Problem 6 on the Fall 2021 AMC 10B contest: *The least positive integer with exactly 2021 distinct positive divisors can be written in the form* $m \cdot 6^k$*, where* m *and* k *are integers and* 6 *is not a divisor of* m*. What is* m + k*?*

I'm reminded of how a triangle can be dissected into n^2 triangles by placing $n - 1$ equally spaced points on each side and connecting the points by line segments parallel to the sides of the triangle. So, 2022 counts the number of triangles in such a pattern with 45^2 triangles less the 3 corner triangles… thus it's the number of triangles in the pattern that are adjacent to at least 2 other triangles.

It's nice to see that a number happens to count the elements of some nice set, although it is generally more fruitful to start with a nice set and then figure out how to count the number of elements it contains. For an example of this, see *On the Number of Tilings of a 4-By-*N *Rectangle with 1-By-1 and 2-By-2 Squares* on page 5.

Say, 3 is a triangular number, since it is 1 + 2. So, 2022 can be represented as the difference between a perfect square and a triangular number. I wonder, what positive integers can be expressed as such a difference? Since perfect squares can be represented as n^2 for some positive integer n and triangular numbers can be represented as $m(m + 1)/2$ for some positive integer m, the question is which positive integers can be expressed in the form

$$n^2 - m(m + 1)/2,$$

where n and m are positive integers?

Curiosity and interest will inevitably allow you to retain more and more mathematical facts over time, such as knowing that $2^{20} = 1{,}048{,}576$, or in this case, recognizing 3 as a triangular number.

Generalization leads to a lot of mathematics. Here, the observation that 2022 is a perfect square minus a triangular number leads

Also, since I know that 2022 can be so represented, how many such representations are there for 2022? That is, how many solutions, in positive integers n and m, are there to the equation

$$n^2 - m(m + 1)/2 = 2022?$$

I really have no idea how to solve this. Maybe using the representation of 2022 as the difference of a perfect square and triangular number will help:

$$n^2 - m(m + 1)/2 = 45^2 - (1 + 2).$$

If I put the 45^2 over to the other side, I'll have a difference of squares that I can factor, and if I put the $m(m + 1)/2$ to the other side, I'll have a difference of triangular numbers, which would be a sum of consecutive numbers – and that should factor, too! Let me try this.

$$n^2 - 45^2 = m(m + 1)/2 - (1 + 2).$$

So,

$$(n + 45)(n - 45) = (m + 3)(m - 2)/2.$$

(To get the right side, I computed the sum of the numbers from 3 to m by adding the first and last, multiplying by the number of numbers from 3 to m, and then halving the result – a standard way of computing sums of arithmetic progressions.)

Hmm. I'm not sure what to do with this equation.

Maybe I can instead try to complete the square in the quadratic $m(m + 1)/2$ and get an equation that equates a difference of squares with a constant.

That is, I know that

$$m(m + 1) = ((m + 1/2) - 1/2)((m + 1/2) + 1/2),$$

so if I let $a = 2m + 1$, then

immediately to the question of which other numbers are, too, as well as the question that Lightning focuses in on: How many ways are there to represent a given number as a perfect square less a triangular number?

Looking for solutions to an equation in positive integers is called **Diophantine Analysis**. Diophantine Analysis has fueled much development in Number Theory, such as with the search for solutions to the famous Fermat equation

$$a^n + b^n = c^n,$$

which was solved by Andrew Wiles in 1994.

How do you think Lightning knew that the difference of triangular numbers should correspond to a factorizable expression?

Here, Lightning succeeds in showing that representations of 2022 as a difference of a perfect square and a triangular number correspond to numbers that can be exhibited as products in two specific ways, namely as

$$2(n + 45)(n - 45)$$

and as

$$(m + 3)(m - 2).$$

This is equivalent to solving the Diophantine equation $2a(a + 90) = b(b + 5)$. Perhaps this equation suggests to you the following related question: Fix two nonnegative integers x and y. What positive integers are both a product of two integers that differ by x and a product of two integers that differ by y?

For certain values of x and y, there are no solutions. For example, if $x = 0$ and $y = 1$, then the question is asking which perfect squares are a product of consecutive integers. Because consecutive integers are relatively prime, for a product of consecutive integers to be relatively prime, both would have to be perfect squares; and the only consecutive

$$m(m + 1) = (a/2 - 1/2)(a/2 + 1/2) = (a^2 - 1)/4.$$

Then, in terms of n and a, the equation becomes

$$n^2 - 45^2 = (a^2 - 1)/8 - 3,$$

or

$$8n^2 - 8(45^2) = a^2 - 25,$$

or

$$8n^2 - a^2 = 8(45^2) - 25.$$

Well, this isn't a difference of squares, but I've seen an equation like this before. I was thinking about what numbers are both perfect squares and triangular numbers, like the number 36, which is both 6^2 and the sum of the numbers from 1 through 8.

I remember that from one solution, we can get infinitely many. And the idea is to consider numbers of the form $a + n\sqrt{8}$ and define the **norm** of the number $a + n\sqrt{8}$ to be

$$(a + n\sqrt{8})(a - n\sqrt{8}) = a^2 - 8n^2.$$

This norm satisfies the beautiful property that the norm of a product of numbers of the form $a + n\sqrt{8}$ is equal to the product of the norms.

The equation that I'm trying to solve, $8n^2 - a^2 = 8(45^2) - 25$, is asking for numbers $a + n\sqrt{8}$ with the same norm as the number $5 + 45\sqrt{8}$. So if I can find a number of the form $a + n\sqrt{8}$ with norm 1, then I can multiply $5 + 45\sqrt{8}$ by this number however many times I wish to obtain other numbers with the same norm – in other words, other solutions to the Diophantine equation!

So what I need to do now is try to find a solution to the equation

$$x^2 - 8y^2 = 1.$$

perfect squares are 0 and 1, leading to a product of 0, which is not positive. On the other hand, we can find values of x and y for which there are solutions by taking any number that can be written as a product in two different ways, such as $6 = 1 \times 5 = 2 \times 3$, and setting x and y to the differences between the factors in the two different products. (This example shows that there are solutions to the case $x = 4$ and $y = 1$.) These considerations lead us to this question: For what x and y does there exist a solution?

Notice that instead of a difference of squares, Lightning obtained a difference of 8 times a square and a square. When you do math, you sometimes need to be flexible and go wherever the math leads.

Here, Lightning recognizes the equation. Today, so much math has been discovered that it's really impossible for any one person to know it all or figure it all out alone. So there's a balance you'll have to find for yourself between how much you want to figure out for yourself and how much you want to learn. Generally speaking, though, the more you know – either by having learned it or by having figured it out – the richer and more fun the subject will become for you.

(Can you prove that the norm of a product is the product of the norms?)

Here, Lightning has stumbled upon a Diophantine equation known as a **generalized Pell equation**. These equations have been well studied and they relate to continued fractions and number fields. The original Pell equation is the Diophantine equation

$$x^2 - Ny^2 = 1,$$

where N is a positive integer.

Here, Lightning boils down the problem to finding a solution to one of the original Pell equations.

Well, there is no solution when $x = 0$, because 1 is not a multiple of 8.

There is the solution $x = 1$, $y = 0$, but that won't help me find other solutions to my original equation because that corresponds to the number 1, and multiplication by 1 doesn't change anything.

There is no solution when $x = 2$, because 1 is not a multiple of 4.

Hey! There is a solution for $x = 3$! In that case, $y = \pm 1$.

So, if I multiply $5+45\sqrt{8}$ by $3+\sqrt{8}$, I should get another solution. I compute that

$$(5+45\sqrt{8})(3+\sqrt{8}) = 375+140\sqrt{8}.$$

So it should be the case that

$$8(140^2) - 375^2 = 8(45^2) - 25.$$

And it is!

Oh, wait, I forgot that I previously made the substitution $a = 2m + 1$, and here, $a = 375$, so $m = 187$. In other words, 2022 is equal to the perfect square 140^2 less the 187th triangular number:

$$2022 = 140^2 - (1 + 2 + 3 + \dots + 187).$$

If I multiply by $3+\sqrt{8}$ one more time, I get

$$2022 = 795^2 - (1 + 2 + 3 + \dots + 1122).$$

I never knew that about 2022!

Here, Lightning finds a solution to the Diophantine equation $x^2 - 8y^2 = 1$ by systematically substituting the first few values of x, then solving for y – and gets lucky!

How would you find a solution to the Pell equation $x^2 - Ny^2 = 1$ if a smallish solution is not available, such as when $N = 109$ (for which the smallest x for which there is a solution to the Diophatine equation

$$x^2 - 109y^2 = 1$$

is

$$x = 158{,}070{,}671{,}986{,}249.$$

Lucky for Lightning that $x^2 - 8y^2 = 1$ has a much smaller solution!)

One aspect that Lightning does not consider is whether *all* solutions to the original Diophantine equation can be found using this method of multiplying by $3+\sqrt{8}$. Can you determine whether that is true or not?

What else can you discover about the number 2022?

If you get the hang of developing your mathematical thoughts, not only will you never be without interesting things to think about, but you will also have a great way to develop your thinking ability since the math will tell you if you've made any errors in logic.

Notes from the Club

These notes cover some of what happened at Girls' Angle meets. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

Session 29 - Meet 9 November 4, 2021	Mentors:	Mandy Cheung, Cecilia Esterman, Jenny Kaufmann, Bridget Li, Tina Lu, Kate Pearce, Vievie Romanelli, Sakshi Suman, Rebecca Whitman, Angelina Zheng

You want to create a list of the numbers 1 through 1000 in a random order, with all possible sequences equally likely. What's a practical and effective way to do that? If you decide to write a computer program to accomplish this, what would your algorithm be?

Session 29 - Meet 10 November 11, 2021	Mentors:	Cecilia Esterman, Jenny Kaufman, Bridget Li, Kate Pearce, AnaMaria Perez, Laura Pierson, Vievie Romanelli, Sakshi Suman, Jane Wang, Rebecca Whitman
	Visitor:	Karia Dibert, University of Chicago

Former Girls' Angle mentor Karia Dibert[3] returned through our Support Network to tell us about her work in astrophysics. Now a graduate student at the University of Chicago, Karia invented a new kind of detector which will be deployed at the South Pole, where it will be used to gather information on the cosmic microwave background radiation. Why the South Pole? Partly because the atmosphere there is very dry, since water freezes out, greatly reducing atmospheric disturbances. In the near future, Karia will be traveling to the South Pole to install the detector. What an exciting journey!

Session 29 - Meet 11 November 18, 2021	Mentors:	Cecilia Esterman, Jenny Kaufman, Bridget Li, AnaMaria Perez, Laura Pierson, Vievie Romanelli, Sakshi Suman, Rebecca Whitman, Angelina Zhang, Rachel Zheng

Can you design a 3D object which casts shadows that look like the different letters of your initials depending on which direction you shine light upon it?

Session 29 - Meet 12 December 2, 2021	Mentors:	Cecilia Esterman, Jenny Kaufman, Bridget Li, Kate Pearce, AnaMaria Perez, Vievie Romanelli, Sakshi Suman, Rebecca Whitman, Angelina Zhang, Rachel Zheng

Due to the pandemic, we ran a regular meet instead of our traditional end-of-session math collaboration. Despite the pandemic, members succeeded in doing a lot of math this semester. Congratulations to all! And many thanks to our mentors for all the wonderful work you did!

[3] Karia also played major, key roles in developing the concept and storyline of SUMIT 2018, 2019, and 2020, although SUMIT 2020 was cancelled due to the ongoing pandemic.

Calendar

Session 29: (all dates in 2021)

September	9	Start of the twenty-ninth session!
	16	
	23	
	30	
October	7	
	14	
	21	
	28	
November	4	
	11	Karia Dibert, University of Chicago
	18	
	25	Thanksgiving - No meet
December	2	

Session 30: (all dates in 2022)

January	27	Start of the thirtieth session!
February	3	
	10	
	17	
	24	No meet
March	3	
	10	
	17	
	24	No meet
	31	
April	7	
	14	
	21	No meet
	28	
May	5	

Girls' Angle has been hosting Math Collaborations at schools and libraries. Math Collaborations are fun math events that can be adapted to a variety of group sizes and skill levels. We will soon have versions available that are designed for remote participation. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

Girls' Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ____________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

The $50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- ☐ Enclosed is a check for $50 for a 1-year Girls' Angle Membership.
- ☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

Girls' Angle

A Math Club for Girls

Girls' Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is $20/meet for members and $30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle is based in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: _______________________
(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is $216 for one session (12 meets)	☐ I will pay on a per meet basis at $30/meet.
☐ I will pay on a per meet basis at $20/meet.	☐ I'm including $50 to become a member, and I have selected an item from the left.

☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____ Date: ________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____________________________________