Girls' Angle Bulletin

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To Foster and Nurture Girls' Interest in Mathematics

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From the Founder

Sometimes, the feeling of being TRAPped is the best feeling in the world! (See page 8.) No, I'm not referring to our pandemic predicament. But the math goes on, and we're looking forward to another session of Girls' Angle, virtual though it sadly has to be. - Ken Fan, President and Founder

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Girls' Angle Bulletin

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This magazine is published six times a year by Girls' Angle to communicate with its members and to share ideas and information about mathematics.

Girls' Angle welcomes submissions that pertain to mathematics.

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Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: *A snail loses the game* by Juliette Majid. For more context, see "3D Trap" on page 8.

An Interview with Sarah Bryant

Cultivate a "growth mindset" by putting in time, practice, and energy to get better at whatever it is you want to do.

Sarah Bryant is a Lecturer in the Mathematics Department at Gettysburg College. She earned her doctoral degree in mathematics from Purdue University under the supervision of Rodrigo Bañuelos.

Ken: I've been eager to interview you because you're interested in biological applications of math, K12 outreach, and gender issues. All three pertain to Girls' Angle, because some of our members are interested in biological applications of math. But first, I'm interested in your journey into mathematics. When did you notice mathematics and what first attracted you to math?

Sarah: My first memories of math were actually the struggles. I struggled with the less than/greater than symbols (I kept thinking "but it is just the same exact thing, said frontwards or backwards"). I also remember my teacher threatening to kick me out of "Gifted" classes since I did not have my multiplication tables down pat. I recall shrinking into my chair, ashamed. I am a first-gen[1] college student, and I lived in poverty until college. Because of our financial (and emotional) instabilities, we moved often. I went to more than 10 schools from kindergarten to 12th grade and had quite a collection of math teachers: from truly awful to inspiring. The "truly awful" included one teacher teasing me for how I "thought" long division worked (actually, I was right). On the other hand, the inspiring teachers (like my AP Calculus teacher in high school) helped me see math as a puzzle and *never* belittled any of their students. My mom was an avid reader but my parents had not taken math past the basics of multiplication and division, so it was my teachers who really brought math into my world. However, I did not initially plan to study math in college, I wanted to be a forensic scientist. But somehow I placed into a high level of math my first year and got hooked!

Ken: That's remarkable. I'm so glad you found your way despite the bad teachers. Was calculus your favorite math topic in K12? What were your favorite topics in college and grad school? What do you like about those topics?

Sarah: I have a fond memory of digging into some problems related to the Witch of Agnesi in precalculus (11th grade). I don't remember the details, just the feeling of having "a-ha" moments sitting at my desk at home, working on the problem set. In college my favorite math topic was real analysis. The great thing about college math is you get to see a variety of topics. When I got to real analysis I felt a mix of comfort and intrigue. Working on those problems was tough but I felt so amazing once I really "got" something. I would even do un-assigned problems. That's when you know it's love. In grad school I started in complex analysis and switched to a subfield that lies in the intersection of probability and analysis. I was drawn once again to the way

[1] That is, Dr. Bryant was the first in her family to attend college. "First-gen" is short for first-generation."

I enjoyed doing the problems and to the approach of my eventual advisor, Dr, Rodrigo Bañuelos, who was an exceptional teacher.

Ken: In your journey to becoming a mathematician, how did you overcome the roadblocks put in your way?

Sarah: I have been incredibly fortunate, but I have also had roadblocks. For one, as I mentioned, I had a wide range of experiences in school. Since my family was often struggling to make ends meet, we moved frequently. Sometimes we moved because we were kicked out of our home for not paying bills, and there were times we weren't sure where we would live. For my entire junior year of high school we lived with a family member, my sister and I sharing a small space in his unfinished basement. In fact, that is where I was working when I had my "a-ha" moment with the Witch of Agnesi! Our struggles meant I was not able to do extracurriculars or prep for college the way some of my friends could. Luckily, I was accepted to Berea College, a college that is specifically for students from economically-disadvantaged backgrounds, and I absolutely loved it. I was a strong student in college and in graduate school I thought I would be able to focus on math and quickly learn all I needed to learn to succeed. I didn't expect the difficulty, both in terms of the actual level of mathematical sophistication and the pain of feeling judged over and over...by faculty, by peers. Even the first day on campus as I was getting my schedule, I was hit with this. I had the same advisor (simply a course advisor, not research advisor) as my then-boyfriend (now-husband) who had gone to the same undergrad. In fact, not to brag, but while he was a very good undergrad student, I was -at least on paper- even better. However, our advisor told him he was well-positioned to take the qualifier-level courses and mere moments later told me I was not.

Ken: What are your qualities that enabled you to be a professional mathematician?

Sarah: In addition to luck, persistence has been an important quality. I have also grown more and more sociable over the years; I have learned to connect more deeply with people, and those connections have created my mathematical world.

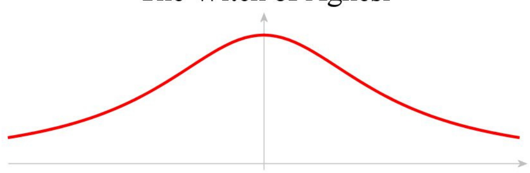
Ken: For aspiring mathematicians, what advice would you give so that they have the best chance of realizing their dream?

Sarah: I'll speak directly to these aspiring mathematicians: I can promise you that things will at times be tough, but you should stay curious, stay open to partnership and collaboration, and stay as optimistic as possible. Cultivate a "growth mindset" by putting in time, practice, and energy to get better at whatever it is you want to do. There is a great big math world out there, and we really want you in it, if it is where you want to be!

Ken: Recently, you participated in a joint program of the Association for Women in Mathematics and the Mathematical Biosciences Institute called Woman Advancing Mathematical Biology where you studied nematocysts, and how jellyfish use their nematocysts to capture prey. How does math play into this topic?

Sarah: This was a fascinating experience, by the way. Until that program I did not know much about jellyfish or the math involved. For our project, we used some sophisticated mathematical tools (and software) to resolve the interactions between fluids and small particles. Our lead researchers had some sense that the

The Witch of Agnesi



The "Witch of Agnesi" is the graph of $y = \dfrac{8a^3}{x^2 + 4a^2}$, where a is a positive constant. The graphs for different values of a are geometrically similar.

extraordinary acceleration of the nematocysts was not simply about puncturing the shells of crustaceous prey (as biologists posited) but perhaps about the interaction with the water. A very small organism (or very small part of a larger one) moves in water much like we might move in syrup. That is, the water is more viscous at this tiny scale. The explosive burst of energy that releases a nematocyst is staggering. We ran several simulations with differing parameters and were able to see that this acceleration in fact must meet these levels (among the greatest in the animal kingdom) to ensure that they will even move through the water to the prey, let alone pierce it.

Ken: What can math bring to biological research in general, and vice versa?

Sarah: During the Woman Advancing Mathematical Biology experience, I saw groups working on projects related to jellyfish, ticks, placentas...so many different areas of biological research. In each group there was a variety of experience with different modeling tools, or different approaches, and math was the common vocabulary underpinning the work. In fact, after a few slides into a presentation from another group, you would just be looking at math formulas and findings. These have some meaning or connection to the biology but in terms of how they are tackled, it looks a lot like pure math! So math brings a lens to solving biological questions and the yet-unanswered biological observations drive math to figure out what must still be missing or incomplete about a model. It is a great back and forth.

Ken: If a girl is interested in both math and biology, are there particular subjects she should study? Are there any examples of mathematical biology that you would recommend that she read about?

Sarah: Amazingly, nearly every kind of pure math has some application and every area of biology has some mathematical models involved...so if a girl is interested in both she will certainly find many opportunities to combine them! We have all lived through a pandemic where mathematics and biology helped push the limits of our expectations for finding and distributing a vaccine to the world...I am certain that there are many more advances waiting, just around the corner, to make our world better. My own background is not in applied mathematics, but as a mathematician I was able to join a group of other math biologists because we have a common training in rigorous thinking, careful reading, and tackling problems using a variety of tools. I had to learn new software and learn about jellyfish, but I understood what was happening in the models, when they didn't make sense, and how to communicate what the predictions meant.

Ken: What are the advantages, if any, of having an all-woman collaboration?

Sarah: Without fail, the all-women-identifying collaborative spaces I have been in have been amazingly supportive. In these spaces, it is expected that we all operate from a sense of assuming the best of everyone, rather than seeking out weaknesses. Even with a great diversity among members in the collaboration, there is an easy camaraderie because of our shared experiences.

Ken: You served as the Executive Director of the EDGE (Enhancing Diversity in Graduate Education) Foundation. What were your goals with EDGE and how can people get involved?

Sarah: The EDGE Program has been running for several years, and I was a participant in 2002 then mentor in 2005 and 2006 and have returned a number of times as an instructor (most recently 2018). As a participant in 2002, I viewed EDGE as a "transition" program that would help me go from a small liberal arts school to an R1 institution[2] with greater confidence and chance of completing the PhD. Over the years, I learned EDGE is much more than a summer program, because the mentoring bonds formed among this incredibly talented and diverse collection of women extends well beyond the first steps into graduate schools.

In 2016, at the time I joined the Foundation, the EDGE Foundation Board was focused on changing the funding model for EDGE and growing the impact of EDGE. Despite the program being lauded as transformative, funding from sources such as NSF has become more difficult over time. Newer programs and large-scale programs are more likely to get the kinds of grants that had helped EDGE operate for years. I worked with the president and other members of the Foundation to investigate new lines of funding, new branding and marketing, and so on. We wanted to find the support that would allow EDGE to not just continue but to change to be even better. EDGE now has a stronger sense of partnership with a variety of institutions and organizations that support women not only in the transition to graduate school but at many other points in their careers.[3]

Ken: What do you think is the underlying reason for the underrepresentation of women in mathematics? What should we be doing to change this?

Sarah: I used to give talks about implicit bias and representation of women in STEM but I do not see myself doing that again. I think the conversations about implicit bias (unconscious attitudes and actions that may lead us to making snap judgments based on stereotypes) have diverted energy from doing what needs to be done to tackle racist outcomes head-on. In other words, these days I am more concerned with concrete actions taken to address underrepresentation and I do not spend as much time trying to convince anyone that racist outcomes do in fact happen in fields like mathematics. This means I have been spending more time on contributing to communities of support and promoting more inclusive policies. If you ask women, and I have, the very real issues of hostility, micro aggressions, withholding of support and mentoring, and sexual harassment are still pushing women out of mathematics or out of leadership roles within mathematics, and it is a shame. We must make these bad behaviors unacceptable; they should not be part of the price women pay for participation in the world of mathematics. We should also ask hard questions of departments where there "just happens" to be few women or a stark lack of diversity among faculty and/or

[2] An "R1 institution" is a category in a classification system of universities managed by Indiana University's Center for Postsecondary Research (but originally created by the Carnegie Foundation for the Advancement of Teaching). An "R1 institution" is a university with a "very high" level of research.

[3] For more about EDGE, please check out https://www.edgeforwomen.org/

students. There has been a long period of time where we talked about the unconscious bias aspect of how these things occur; by now, people know that their actions may not seem overtly sexist or racist but could still be so. Now let's move to the next step: let's hold ourselves accountable to make things better.

Ken: You co-created the Shippensburg Area Math Circle[4] for 4th and 5th graders. What motivated you to work with younger children?

Sarah: My husband and I are both mathematicians and have had so much fun playing games and investigating math with our daughter (now 13). We thought engaging in some casual after school math enrichment would be a fun way to involve more kids. Once we connected with more people doing similar work, we were swept up in the enthusiasm and joy of the math circle community.

Ken: Could you please share with us a story or two from your work with students at the Shippensburg Area Math Circle?

Sarah: I have so many great memories of our work, and once we are able to resume more normal operations I desperately want to re-start the Circle (down since Covid). Some of my favorites were our cryptography mysteries, with multi-step clues taking students all around the math building. I doubt the halls often ring with that much shrieking and laughter. I also have many, many moments with kids who at some point "lit up" during a session, whether it was making interlocking Möbius valentine hearts or solving logic riddles or using the power of fractions to untwist dancing tangles. Year after year we had lovely Saturdays...the three of us Bryants moving the chairs and desks, setting up our supplies, eager to bring our math play beyond our kitchen table to share with others.

Ken: You do so many wonderful and good things. I have a really basic question for you about that: What motivates you?

Sarah: Very flattering, but it's so simple. As I said at the beginning, over the years I have become more and more open to collaborative experiences and going outside my comfort zone. This has landed me in some amazing places. Because of my life's journey, especially through poverty and hardships, I have a hard time saying no to any opportunity. I feel so grateful to be here (in the "math world") and maybe I am still trying to prove that I actually belong.

Ken: What do you like to do when you're not doing math or math outreach?

Sarah: I like to read, solve puzzles and crossword puzzles, try new arts and crafts with my daughter, and recently started some fitness routines that are fun.

Ken: Do you have any parting advice for our readers?

Sarah: I can't wait to see what you are going to do for this world, and I hope that you are able to go through your math journey without some of the same roadblocks others have faced. But if you run into one, please forgive us for not clearing them all out of your way and do your best to make the path even better for those who will come after you.

Ken: Thank you for this interview!

[4] See this presentation by Dr. Bryant on YouTube: https://www.youtube.com/watch?v=JigcKohxi2Y

3D Trap

by Eva Arneman, Altea Catanzaro, and Saideh Danison
edited by Rebecca Whitman and Amanda Galtman

Primary guidance and mentorship for this mathematical investigation was provided by Rebecca Whitman.

Title logo designed by Altea Catanzaro.

Introduction

3D Trap is a game that is seemingly of luck at first, but as time tells, is of strategy.

We have found that when the game board is a tetrahedron, the first player can always win, no matter what the second player does! Down below are our reasons. We also include a manual to best strategize your gameplay. The manual below is for the tetrahedron, while the rules can be applied to the edges and vertices of any polyhedron or graph. Enjoy!

Rules

The pinnacle of this whole operation.

Setup: The first step is to draw a 3D shape. You can also play on a combinatorial graph of vertices and edges. To begin, have each player pick a color. We suggest you play with two players, but you can experiment with more.

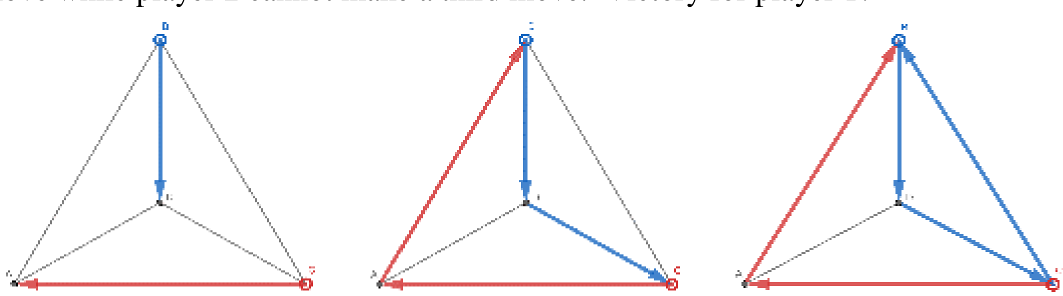
How to win: To win, you must make the other player run out of moves and make the last move yourself.

How to play: After you have set up, player 1 picks a vertex and draws a small colored circle where they start. This is important so they can remember where they started and which way they are going. Player 1 then colors a line from their chosen vertex along an edge until they hit the first vertex in their path. Player 2 does the same, using a different color. Players can revisit vertices but cannot reuse an edge that any player has colored.

Next, player 1 makes another straight line from the vertex they ended at in their last turn, until they hit another vertex. Player 2 should also follow these steps.

Continue the steps above until a player has been TRAPped. Being trapped means all the lines directly around the player have been drawn on, which leaves them with nowhere to legally move.

Here is an example of a game. Each figure shows the game after both players make a move. Player 1 is shown in blue, and player 2 in red. You can see that player 1 is able to win on their third move while player 2 cannot make a third move. Victory for player 1!



The First Player's Handbook

This is a strategy guide for the first player to follow on the tetrahedron. We will prove that this strategy always works.

The first step is to pick any vertex to begin on, make your circle, and follow a line from that circle to another vertex. This move does not matter. After the second player makes their move, start to make a triangle. There are two possibilities for what to do.

Possibility 1: The easier possibility. If the other player has not touched your starting or ending vertex, then go toward their starting point. This is what happened in the example game, above.

Possibility 2: If the other player has touched your starting or ending vertex, go to the only uncolored vertex.

The third step is also a split step.

Possibility 1: If you can finish your triangle since that line is not taken, do so!

Possibility 2: If the second player is out of moves and you have already won, then you can stop here or you can finish the triangle by going to your starting position.

Possibility 3: If the last line of your triangle is unfortunately taken, then you should first cut off the other player by going on the only uncolored line coming out of your current vertex. After winning, if you want to make a triangle, go to the last uncolored line.

The last two possibilities include extra moves after the game is over and you win. They are written purely for evidence of our proof that we will explain later.

The last and best step is to relish the victory!! YAY!

The Second Player's Handbook

Prepare to lose, and accept your loss. Try playing on a cube next time or persuade the other person to make you go first.

Theorem

On the tetrahedron, the first player can always make a triangle, and this enables them to win. One way to do this is to use the first player's handbook.

Proof

The Triangle approach!

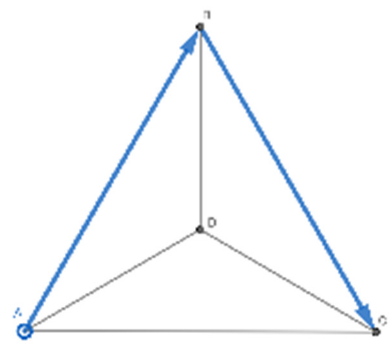
First, we will prove that player 1 can always make a triangle if they are following the first player's handbook. Then, we will show that if the first player finishes a triangle, they must be the winner.

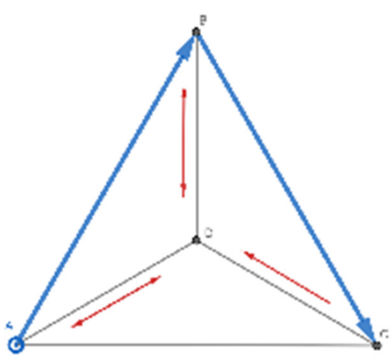
Following the first player's handbook is essential to make this proof work.

To explain this, we will use the figure below, where the vertices are labeled A, B, C, and D. Player 1's starting vertex is circled in blue, and their edges are blue arrows. Player 2's starting vertex and edges will be in red.

We will start in the middle of a game. We can see what the first player's moves are, but we don't know what the second player has done. We will work backwards to figure out what the second player's first move was, then work forward to show that the first player can finish a triangle. We also assume that the first player is following the first player's handbook. This assumption means that while many moves are possible for the second player, certain moves could not have happened because they are not consistent with the first player's next move.

In all of the scenarios below, the first player goes from A to B. After that, the second player makes an unknown move, which provokes the first player to make their second move in response, following the first player's handbook. This move is from B to C. Since a tetrahedron is symmetric on all sides (even though its 2D projection doesn't look so!), we are free to label the vertices so that the first player's moves are as we state.

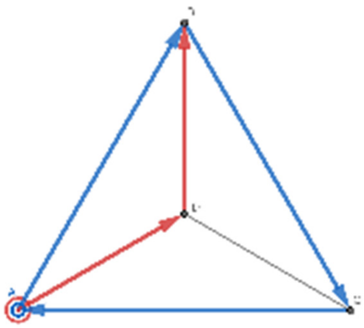




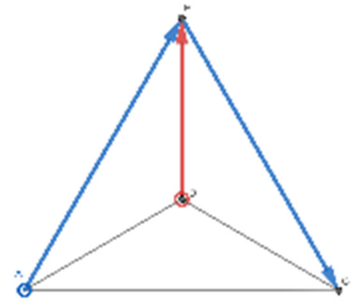
What could player 2's first move have been? There are five possible moves that player 2 could have made. These are D to A, A to D, D to B, B to D, and C to D. In all of these scenarios except C to D, the first player uses Possibility 2 for their second move in the first player's handbook and goes to the uncolored vertex (B to C). If the second player went from C to D, then the first player uses Possibility 1 for their second move, to go to player 2's starting vertex (B to C). The second player's first move could NOT have been A to C, C to A, or D to C, because then the first player's second move from B to C would not have followed the first player's handbook.

What could player 2's second move be? What could player 1's third move be?

Now that we know all the moves that player 2 could have started with, we will look at all the possibilities for player 2's second move and player 1's third move.

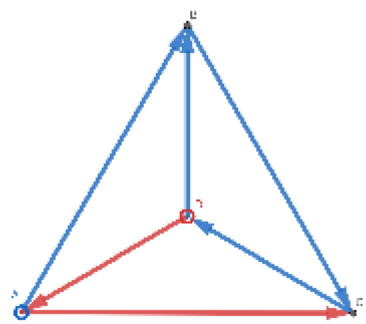


If the second player started with C to D, B to D, or A to D, then no matter what their second move is, player 1 is able to finish a triangle on move 3. That traps player 2, so player 1 wins.

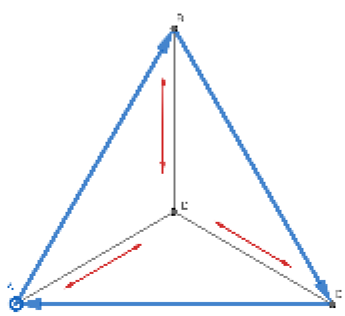


If the second player started with D to B, then they will not have a second move, and the first player will be able to make a triangle on their 3rd turn as well. This would follow possibility 2 of the first player's handbook.

The only other starting option for the second player is D to A. The move D to A observes the 3rd possibility in the first player's handbook, in which the 1st player makes a triangle in four moves but still wins. Player 1's first move is A to B, the second move is B to C, the third move is C to D, and the fourth and final move is D to B. We also have a diagram for this scenario below. Please keep in mind that player 2's second move does not matter in this scenario.



In conclusion, if the first player follows the first player's handbook and makes a triangle, then they are guaranteed a win!



Are triangles useful for the second player? No. You see, when you make a triangle, you use up one side of every other triangle. This causes the second player to be unable to make a triangle, because to make a triangle, you need all three sides. This is crucial to the first player's strategy, but the second player can never do this. By the end of the game, the first player will always have executed three moves (not including added moves after winning) and the second player only two moves, so the first player is the only one capable of making a triangle. In any case, the first player is the only one who wins with the triangle strategy.

Conclusion

In conclusion, on a tetrahedron in the game of 3D Trap, there is a clear strategy that the first player can use to win. The triangle strategy is a simple and always effective way to play. Thank you for taking the time to read our theorem and proof!

We would be delighted to see if you make a theorem of your own on the tetrahedron or another shape. Thank you, again!

The Needell in the Haystack[1]

Geometry, Intuition, and the Kaczmarz Algorithm
by Anna Ma | edited by Jennifer Sidney Silva

Information is all around us; it always has been. Recent advances in technology have given us access to this information in the form of data. Before Apple watches and Fitbits, continual monitoring of heart rate in beats per minute was not easily accessible to the mainstream consumer. This information was only procurable when at the doctor's office or at a hospital when large, wired EKG machines were hooked up to a person. Now, we can access that information in the form of data simply by lifting our wrists.

The beauty of having access to this data is that we can answer so many questions that once went unanswered: What's my resting heart rate and how does it compare to the heart rate of a healthy person? What's my heart rate when I'm feeling sad versus when I'm feeling angry? How high does my heart rate get before a test? (I have actually monitored my heart rate on the day of a big presentation, and it reached 140 beats per minute just before the talk!) While most of the aforementioned questions can be answered with simple statistics (such as taking the maximum or average over a small sample set), there are more complex questions that require vast amounts of data. For example, given a continuous stream of heart-rate data, can we classify what activity a person is performing (lying down, walking upstairs, etc.)? As another example, can we infer how many steps a person took based on 24-hour heart-rate data?

When it comes to training an algorithm to answer any of these questions, computational challenges quickly come to light. What if the data set is so large that we cannot load it into our computer's local memory? What if we don't have access to the entire data set, but only to small portions of it at a time (because it's streaming in over time)? Classical algorithms for answering these types of questions were not designed with computational constraints in mind. This has led to the design, analysis, and the still ever-growing body of research in randomized numerical linear algebra.

In the Volume 14, Number 3 installment of "Needell in the Haystack," Prof. Needell explained the randomized Kaczmarz algorithm and looked closely at how it behaves when a linear system is consistent. In this article, we revisit the randomized Kaczmarz algorithm through the lens of different types of linear systems and highlight how the geometry of a problem can drive our intuition of how the Kaczmarz algorithm works in different settings. First, we will review and lay out the groundwork for understanding the geometry of the problem at hand. Next, we will reintroduce the Kaczmarz algorithm and investigate how the algorithm is expected to perform in different problem settings.

Problem Formulation

Before we can jump into the specifics of the randomized Kaczmarz algorithm, we must first take our data and formulate it as a mathematical problem. To that end, let's consider the following: suppose we're given data in the form of n-dimensional real vectors a_i, for $i = 1, \ldots, m$, where each data point has a corresponding **measurement** $y_i = \langle a_i, x\rangle$. Here, x is an unknown signal vector that we want to find and $\langle a_i, x\rangle$ is called the **inner product** and stands for $\sum_{j=1}^{n} a_{ij}x_j$, where a_{ij} is the j^{th} component of the vector a_i and x_j is the j^{th} component of the signal vector x. To put this model into context, consider the example where we want to predict the number of

[1] This content supported in part by a grant from MathWorks. Anna Ma is a former student of Deanna Needell's.

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Ken Fan
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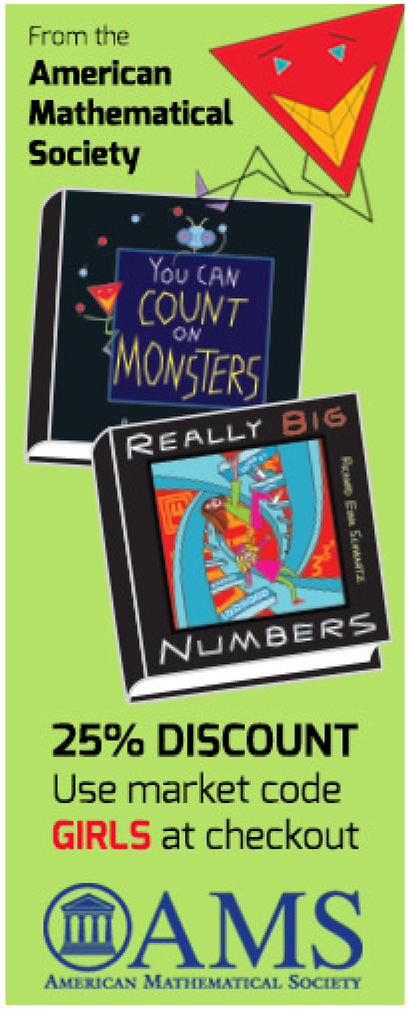

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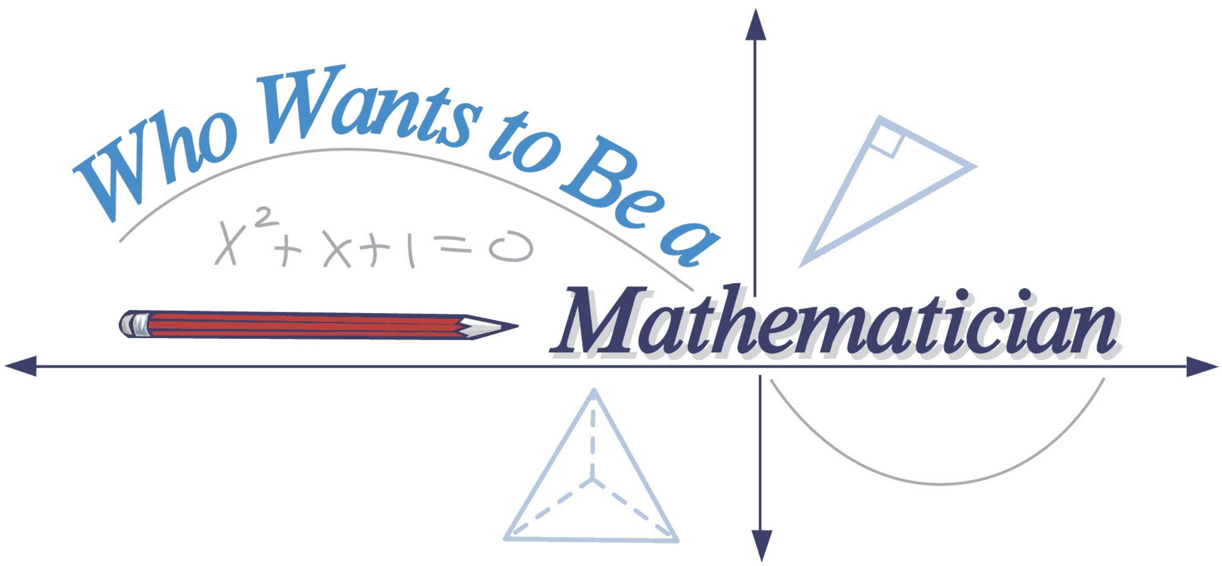

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Order In Disorder 2:

Uncovering the Mathematical Rules of Crumpling
by Jovana Andrejevic[1] | edited by Amanda Galtman

The physics behind the math

Part 1 of this article introduced a surprising experimental result about crumpling: A thin sheet of Mylar crumpled repeatedly inside a cylindrical piston exhibits logarithmic growth in the total length of creases, referred to as the sheet's mileage. The robustness of the logarithmic scaling of mileage piqued the interest of collaborator and applied mathematician Prof. Chris Rycroft and his group. Specifically, they wondered what physical principles could bring about this behavior. The mileage formula relating total crease length to the number of times crumpled was an *empirical* result, based only on experimental observation rather than theory. However, in 2021, the computational and experimental teams proposed a possible physical explanation for the origin of the logarithm, lending the missing theory to the model [4]. They likened crumpling to fragmentation, the process of breakup into smaller and smaller pieces that dictates how rocks crumble into pebbles, or glass shatters into fragments. While crumpled sheets always remain intact, the partitioning of their surface into smaller and smaller facets over time proved consistent with fragmentation theory. This theory gave researchers the new framework they needed to derive the logarithmic scaling on physical grounds.

Can the rules of flat-folding elucidate crumpling?

There is still so much we don't fully understand about crumpling, and physicists and mathematicians continue to pursue the hidden rules behind this complex process. By contrast, the field of *flat-folding* has rich theoretical developments in geometry, combinatorics, and graph theory, to name a few. Rigid flat-folding is the process of repeatedly folding a sheet along straight lines such that the resulting polygonal shape always remains flat, as in Figure 4.

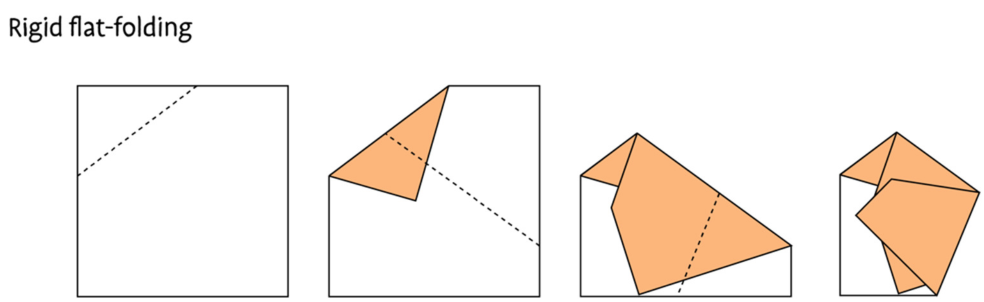

Figure 1: **Rigid flat-folding.** Sample process of folding a sheet such that it always remains flat in the plane.

In contrast to crumpling, rigid flat-folding follows strict geometric rules. Two well-known rules that apply locally at any vertex are:

[1] This work is supported in part by a grant from MathWorks.

1. **Maekawa's theorem** [5]: The number of mountain and valley folds that meet at a single vertex must differ by two. A consequence of this theorem is that the number of creases that meet at a vertex must be even.

2. **Kawasaki's theorem** [6]: Alternatingly adding and subtracting the angles around a vertex must come out to zero.

These rules are illustrated in Figure 5.

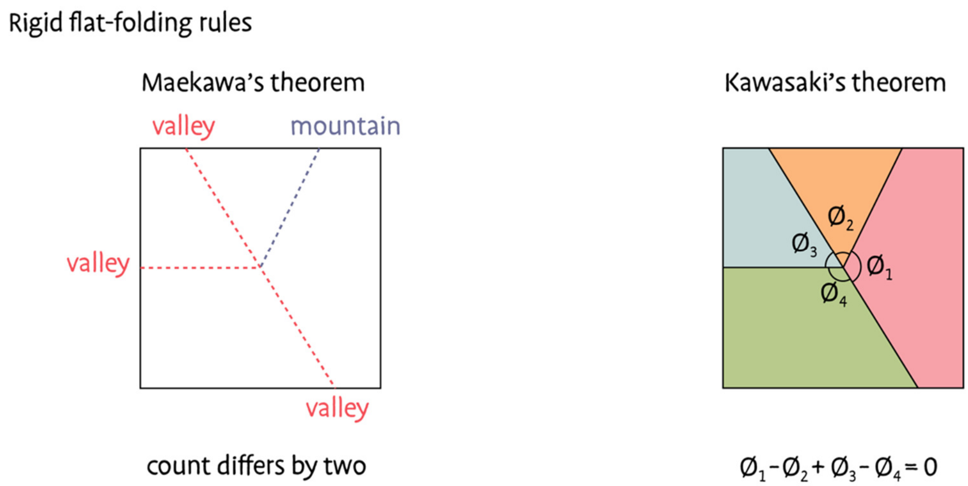

Figure 5: **Rigid flat-folding rules**. Two rigid flat-folding rules: Maekawa's theorem and Kawasaki's theorem.

A crumpled sheet that is compressed enough will approach a flat-folded state, so it's natural to wonder whether flat-folding could shed some light on crumpling. In 2019, the formidable team of Profs. Rubinstein and Rycroft and their students published a study that did just that! Using a growingly popular data analysis technique, machine learning, the groups investigated a problem of pattern reconstruction [7]: Given only the locations of valley folds, can we predict the locations of the mountain folds? The key to their approach was to combine high quality experimental crumpled crease patterns (complex) and computer-simulated flat-folding patterns (simple) into one dataset composed of hundreds of images. The team trained a neural network – a type of machine learning model – to predict the distance to the nearest mountain fold at every point in space. If the network predicts a distance close to zero at a particular point in space, we expect there should be a mountain fold nearby. If it predicts a large distance, we do not expect a mountain fold close by. The basis of the study is illustrated in Figure 6.

The researchers discovered that their machine learning model was highly successful at reconstructing the mountain folds of flat-folded patterns. While it could not recover the crumpled patterns as accurately, it was able to capture some key features, especially in areas very near or very far from a mountain crease. Even more fascinating, they discovered that if they deliberately "tricked" the model by training on simulated flat-fold patterns that violated Maekawa's and Kawasaki's theorems, the model in turn made poorer predictions about crumpling as well. This outcome hints that there is indeed some hidden geometric order to crumpling patterns and brings us one step closer to untangling these beautiful, complex networks through mathematical approaches.

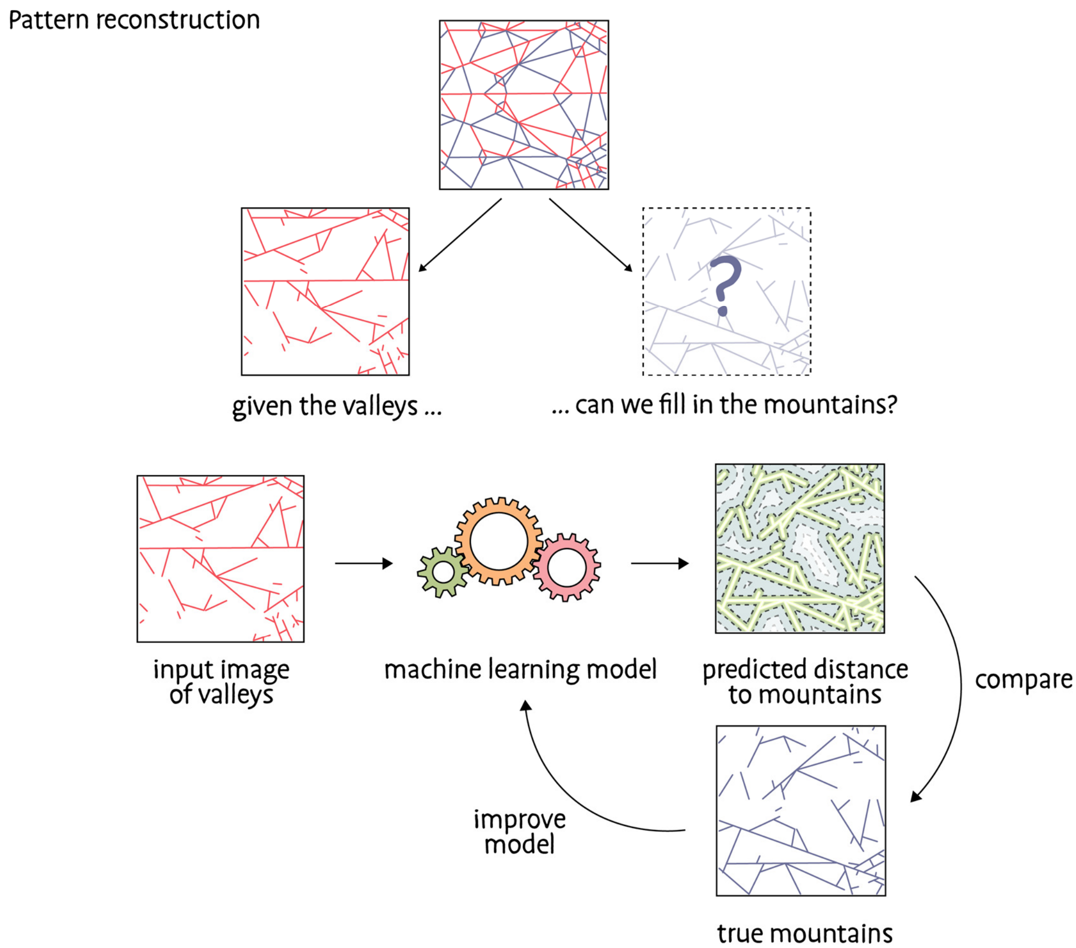

Figure 2: **Pattern reconstruction problem.** A crease pattern can be split into valley and mountain folds, prompting the question of whether the mountain folds, for instance, can be inferred from the valleys. Predictions of corresponding mountain fold locations were made by training a machine learning model on hundreds of example images. During the training process, an image of valleys is supplied as input to the model, which outputs a prediction of the distance to the nearest mountain folds. The image of true mountain folds is used to calculate the prediction error and incrementally update the model parameters to improve the prediction.

References

[4] Andrejevic, J., Lee, L. M., Rubinstein, S. M., and Rycroft, C. H. (2021). A model for the fragmentation kinetics of crumpled thin sheets. *Nature communications*, 12(1), 1-10.

[5] Barile, Margherita. "Maekawa's Theorem." From MathWorld--A Wolfram Web Resource, created by Eric W. Weisstein. https://mathworld.wolfram.com/MaekawasTheorem.html

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[7] Hoffmann, J., Bar-Sinai, Y., Lee, L. M., Andrejevic, J., Mishra, S., Rubinstein, S. M., and Rycroft, C. H. (2019). Machine learning in a data-limited regime: Augmenting experiments with synthetic data uncovers order in crumpled sheets. Science advances, 5(4), eaau6792.

Summer Fun!

In the previous issue, we presented the 2021 Summer Fun problem sets.

In this issue, we give solutions to many of the problems.[†] Our solutions may be terse and, in some cases, are more of a hint than a solution. We prefer not to give detailed solutions before we know that most of the members have spent time thinking about the problems. The reason is that *doing* mathematics is very important if you want to learn mathematics well. If you haven't tried to solve these problems yourself, you won't gain as much when you read these solutions.

If you haven't thought about the problems, we urge you to do so *before* reading the solutions. Even if you cannot solve a problem, you will benefit from trying. By working on the problem, you will force yourself to think about the associated ideas. You will gain familiarity with the related concepts and that will make it easier and more meaningful to read other's solutions.

With mathematics, don't be passive! Be active!

Move your pencil and move your mind – you might discover something new.

Also, the solutions presented are *not* definitive. Try to improve them or find different solutions.

Solutions that are especially terse will be indicated in red. Please do not get frustrated if you read a solution and have difficulty understanding it. If you run into difficulties, we are here to help! Just ask!

Members: *Don't forget that you are more than welcome to email us with your questions and solutions!*

Please refer to the previous issue for the problems.

[†]We're holding off on solutions to Fan Wei's problem set to respect a subscriber's request.

Summer Fun!

Wythoff’s Game

by Laura Pierson

In these solutions, we will refer to (b, a) as the “mirror of (a, b)”.

1. Notice that no game can end in a draw. Somebody must win. For a given starting position, if there is no way for the first player to play in order to guarantee a win, it means that the second player has a way to win. Since the mirror image of every legal move is also a legal move, if (a, b) is a winning position, the first player can play a mirror strategy to guarantee winning the starting position (b, a).

2. From (n, n), the first player can immediately win by taking n from both piles. Similarly, the first player can immediately win from $(0, n)$ by taking everything from the second pile.

3. From $(1, 2)$, the only options are to take 1 from the first pile, take 1 or 2 from the second pile, or take 1 from both piles. These result in the positions $(0, 2)$, $(1, 1)$, $(1, 0)$, or $(0, 1)$, respectively. The second player can immediately win from any of these positions.

4. The position $(1, n)$, for $n > 2$, can be turned into a losing position by taking $n – 2$ from the second pile and leaving the other player with $(1, 2)$. By Problem 3, from $(1, 2)$ the second player will lose, so the first player wins. Similarly, from $(2, n)$, for $n > 1$, the first player can take $n – 1$ from the second pile and leave the second player with $(2, 1)$, which is equivalent to $(1, 2)$. Finally, from $(n, n + 1)$, the first player can take $n – 1$ from both piles and leave the second player with the losing position $(1, 2)$.

5. From $(3, 5)$, taking from the first pile could result in $(0, 5)$, $(1, 5)$, or $(2, 5)$, taking from the second pile could result in $(3, 0)$, $(3, 1)$, $(3, 2)$, $(3, 3)$, or $(3, 4)$, and taking from both piles could result in $(0, 2)$, $(1, 3)$, or $(2, 4)$. Each of these positions falls into one of the categories from either Problem 2 or Problem 4, which means the second player can win from any of those positions.

6. Using the same idea as in the solution to Problem 4, the first player can reduce any of these positions to $(3, 5)$. By Problem 5, there is no way for the second player to force a win from $(3, 5)$, so the first player can win.

7. If the first player can reduce (m, n) to some losing position, then the second player will be left with that losing position. Since the second player is now starting from a losing position, there is no way for them to force a win and so the first player will win under optimal play.

Summer Fun!

8. If there is no way to get to a losing position from (m, n), then the first player will be forced to leave the second player with a winning position. (Recall from Problem 1 that every position is either a winning position or a losing position.) Thus, the second player will have a winning strategy no matter what the first player does, so the first player will lose under optimal play and (m, n) is a losing position.

9. From Problems 2-6, the only losing positions where either pile has size 1, 2, 3, or 5 are (1, 2), (3, 5), and their mirrors. From (4, 7), it isn't possible to get to either (1, 2) or (3, 5) in a single move, so any resulting position with a pile size of 1, 2, 3, or 5 would be a winning position. The only positions you could get to that don't have either of these pile sizes are (4, 4) and (4, 6), but these are also winning positions since (4, 4) has two equal piles and (4, 6) can be reduced to (3, 5). Thus, there is no way to get to a losing position from (4, 6), so it must itself be a losing position. By the ideas from Problem 4 and Problem 6, $(4, n)$ is a winning position for $n > 8$ and $(7, n)$ and $(n, n + 3)$ are winning positions for $n > 4$ because (4, 7) can be reached from those positions.

Similarly, from (6, 10), the losing positions (1, 2), (3, 5), (4, 7), and their mirrors, cannot be reached, and the only way to avoid creating a pile of size 1, 2, 3, 4, 5, or 7 is to go the position (6, 6), (6, 8), or (6, 9). These can be reduced to (0, 0), (3, 5), and (4, 7), respectively, and so are all winning positions. Thus, all the positions that can be reached from (6, 10) are winning positions, so (6, 10) is a losing position. The positions $(6, n)$ for $n > 10$ and $(10, n)$ and $(n, n + 4)$ for $n > 6$ are winning positions because they can be reduced to (6, 10).

10. If there were two losing positions with a common entry or with the same difference, the larger one could be reduced to the smaller one and so would have to be a winning position, a contradiction.

To show that every number and every difference must show up at some point, we proceed inductively. So far, we have found the losing positions (1, 2), (3, 5), (4, 7), (6, 10), and their mirrors. Let m be the smallest number not yet seen in a losing position and let d be the smallest difference not yet seen in a losing position. Consider the position $(m, m + d)$. We claim that there is no way to get from there to a losing position, and so this position must be the next losing position.

Taking from both piles would result in a smaller position with the same difference d, which must be a winning position because d is the smallest difference not yet seen in a losing position.

Taking from the first pile would result in a position with difference more than d and a pile of size less than m. Such positions are all winning positions because m is the smallest pile size not yet seen in a losing position, so each of these can be reduced to some losing position.

Taking from the third pile would result in either a difference smaller than d or a pile size smaller than m. Either way we get something which can be reduced to a losing position, because all pile sizes less than m and all differences less than d show up in some smaller losing position.

Summer Fun!

11. After (1, 2), (3, 5), (4, 7), and (6, 10), the smallest number we haven't used yet is 8, and the smallest difference we haven't used yet is 5, so the next losing position is (8, 13). After that, the smallest number not used is 9 and the smallest difference is 6, so the next losing position is (9, 15). Continuing in this manner, the next few losing positions are (11, 18), (12, 20), (14, 23), (16, 26), and (17, 28).

12. The golden ratio is defined to be the length such that if a unit square is removed from a φ by 1 rectangle, leaving a $\varphi - 1$ by 1 rectangle, the two rectangles have the same aspect ratio. That is, φ must satisfy $\varphi/1 = 1/(\varphi - 1)$. Cross multiplying gives $\varphi(\varphi - 1) = 1$. Expanding and rearranging gives $\varphi^2 - \varphi - 1 = 0$.

13. The quadratic formula gives $\varphi = \frac{1 \pm \sqrt{5}}{2}$. Since φ must be positive, we get $\varphi = \frac{1+\sqrt{5}}{2}$. Plugging this into a calculator gives the approximation $\varphi \approx 1.618\ldots$.

14. First, we show that each positive integer appears in at least one of the two sequences. Instead, suppose that the positive integer n appears in neither sequence, so $\lfloor ar \rfloor < n < \lfloor (a+1)r \rfloor$ for some positive integer a and $\lfloor bs \rfloor < n < \lfloor (b+1)s \rfloor$ for some positive integer b. Then

$$ar < n < (a + 1)r - 1 \quad \text{and} \quad bs < n < (b + 1)s - 1.$$

The inequalities are strict because r and s are irrational. Dividing the first inequality by r and the second by s gives

$$a < n/r < a + 1 - 1/r \quad \text{and} \quad b < n/s < b + 1 - 1/s.$$

Adding the two inequalities and using the fact that $1/r + 1/s = 1$ gives

$$a + b < n < a + b + 1.$$

But this means that n is strictly between two consecutive integers, which isn't possible. Therefore, every positive integer must appear in at least one of the two sequences.

Next, we show that no number appears in both sequences. Instead, suppose that $n = \lfloor ar \rfloor = \lfloor bs \rfloor$, for some positive integers a and b. Then we have

$$ar - 1 < n < ar \quad \text{and} \quad bs - 1 < n < bs.$$

The inequalities are strict because r and s are irrational. Dividing the first inequality by r and the second by s gives

$$a - 1/r < n/r < a \quad \text{and} \quad b - 1/s < n/s < b.$$

Summer Fun!

Adding the two inequalities and using the fact that $1/r + 1/s = 1$ gives

$$a + b - 1 < n < a + b.$$

Again, n is sandwiched between two consecutive integers, which is impossible. This proves that every positive integer shows up in exactly one of the two sequences.

15. From Problem 12, φ satisfies $\varphi^2 - \varphi - 1 = 0$. Rearranging gives $\varphi + 1 = \varphi^2$. Dividing by φ^2 gives $1/\varphi + 1/\varphi^2 = 1$, as desired.

16. The first few terms of the sequences are

$$\lfloor \varphi \rfloor = \lfloor 1.618\ldots \rfloor = 1\,,\ \lfloor 2\varphi \rfloor = \lfloor 3.236\ldots \rfloor = 3\,,\ \lfloor 3\varphi \rfloor = \lfloor 4.854\ldots \rfloor = 4\,,\ \ldots$$

and

$$\lfloor \varphi^2 \rfloor = \lfloor 2.618\ldots \rfloor = 2\,.\ \lfloor 2\varphi^2 \rfloor = \lfloor 5.236\ldots \rfloor = 5\,,\ \lfloor 3\varphi^2 \rfloor = \lfloor 7.854\ldots \rfloor = 7\,,\ \ldots$$

The terms in the first Beatty sequence are the smaller entries in the first few losing positions (1, 2), (3, 5), and (4, 7), and the terms in the second sequence are the larger entries in those losing positions!

17. Since $\varphi^2 = \varphi + 1$, we get $\lfloor n\varphi^2 \rfloor = \lfloor n(\varphi+1) \rfloor = \lfloor n\varphi + n \rfloor = \lfloor n\varphi \rfloor + n$. Thus, $\lfloor n\varphi^2 \rfloor - \lfloor n\varphi \rfloor = n$.

18. We already know the first few terms work by Problem 16. From there we can use induction. After the first $n - 1$ terms, we've seen the differences 1, 2, 3, …, $n - 1$, so the next term should have difference n. By Problem 15, the Beatty sequences for φ and φ^2 are complementary, so each positive integer shows up in exactly one of the two sequences. Thus, the next smallest number not in either entry of any of the previous losing positions

$$(\lfloor \varphi \rfloor, \lfloor \varphi^2 \rfloor),\ (\lfloor 2\varphi \rfloor, \lfloor 2\varphi^2 \rfloor),\ (\lfloor 3\varphi \rfloor, \lfloor 3\varphi^2 \rfloor),\ \ldots,\ (\lfloor (n-1)\varphi \rfloor, \lfloor (n-1)\varphi^2 \rfloor)$$

must occur in exactly one of the two Beatty sequences. Since $\varphi < \varphi^2$, the smallest term we haven't seen yet must be $\lfloor n\varphi \rfloor$. Thus, by Problems 10 and 17, the next losing position should be $(\lfloor n\varphi \rfloor, \lfloor n\varphi \rfloor + n) = (\lfloor n\varphi \rfloor, \lfloor n\varphi^2 \rfloor)$.

Summer Fun!

Parting Ways

An Exploration of Fibonacci Partitions

by AnaMaria Perez and Josh Josephy-Zack

1.

n	5	6	7	8
$p(n)$	7	11	15	22

2. A.

n	1	2	3	4	5	6	7	8	9	10	11	12
f_n	1	1	2	3	5	8	13	21	34	55	89	144

B.

n	f_n	Fibonacci partitions (r)
1	1	(1)
2	1	(1)
3	2	(2), (1+1)
4	3	(3), (2+1)
5	5	(5), (3+2), (3+1+1)
6	8	(8), (5+3), (5+2+1)
7	13	(13), (8+5), (8+3+2), (8+3+1+1)
8	21	(21), (13+8), (13+5+3), (13+5+2+1)
9	34	(34), (21+13), (21+8+5), (21+8+3+2), (21+8+3+1+1)
10	55	(55), (34+21), (34+13+8), (34+13+5+3), (34+13+5+2+1)
11	89	(89), (55+34), (55+21+13), (55+21+8+5), (55+21+8+3+2), (55+21+8+3+1+1)
12	144	(144), (89+55), (89+34+21), (89+34+13+8), (89+34+13+5+3), (89+34+13+5+2+1)

C.

n	1	2	3	4	5	6	7	8	9	10	11	12
f_n	1	1	2	3	5	8	13	21	34	55	89	144
r_n	1	1	2	2	3	3	4	4	5	5	6	6

It appears that $r_1 = r_2 = 1$ and there is the recursive formula $r_n = r_{n-2} + 1$, for $n > 2$.

Summer Fun!

3. A.

n	**Fibonacci partitions (r)**
1	(1)
2	(2), (1+1)
3	(3), (2+1)
4	(3+1), (2+1+1)
5	(5), (3+2), (3+1+1)
6	(5+1), (3+2+1)
7	(5+2), (5+1+1), (3+2+1+1)
8	(8), (5+3), (5+2+1)
9	(8+1), (5+3+1), (5+2+1+1)
10	(8+2), (8+1+1), (5+3+2), (5+3+1+1)
11	(8+3), (8+2+1), (5+3+2+1)
12	(8+3+1), (8+2+1+1), (5+3+2+1+1)

B. Yes. Since $f_{n+1} - f_n = f_{n-1}$ and $f_{n-1} < f_n$ (with the exception of $f_1 = f_2 = 1$), every integer between f_n and f_{n+1} can be written as a Fibonacci partition by adding f_n to the Fibonacci partition of any integer up to f_{n-1}.

C. Yes. We prove this using an inductive argument adapted from Wikipedia.†

Base Case: The number 1 has the Fibonacci partition (1).

Inductive hypothesis: Suppose all positive integers less than some arbitrary positive integer n have Fibonacci partitions.

Inductive Step: We will approach this using cases. For any positive integer n, it follows that n either is or is not a Fibonacci number. In the first case, where n is a Fibonacci number, a Fibonacci partition of n is merely (n). In the second case, where n is not a Fibonacci number, it follows that there exists some integer j such that $f_j < n < f_{j+1}$.

Let a be defined as the distance between n and the nearest Fibonacci number less than n, i.e., $a = n - f_j$. By this definition, $a < n$, therefore by our inductive hypothesis, a has a Fibonacci partition. Moreover, since $a < f_{j+1} - f_j = f_{j-1}$, any Fibonacci partition of a will consist only of Fibonacci numbers less than f_{j-1}. Therefore, since $f_j > f_{j-1}$, we can add any Fibonacci partition of a to f_j to obtain a partition of n that does not repeat any Fibonacci numbers (except for possibly 1, which is allowed to appear twice), and, hence, is a valid Fibonacci partition of n.

4. Answers will vary for parts A, B, and C.

D. Yes, this is always possible. We will prove this using induction, adapting a proof from Wikipedia.†

†See the proof at https://en.wikipedia.org/wiki/Zeckendorf%27s_theorem.

Summer Fun!

Base case: The number 1 has the Fibonacci partition (1).

Inductive hypothesis: Suppose all positive integers less than some arbitrary positive integer n have **Zeckendorf representations**, i.e., can be split up into non-consecutive Fibonacci numbers.

Inductive step: We will approach this using cases. For any positive integer n, it follows that n either is or is not a Fibonacci number. In the first case, where n is a Fibonacci number, the Zeckendorf representation of n is (n). In the second case, where n is not a Fibonacci number, there exists some integer j such that $f_j < n < f_{j+1}$.

Let a be defined as the distance between n and the nearest Fibonacci number less than n, i.e., $a = n - f_j$. By this definition, $a < n$, therefore by our inductive hypothesis, a has a Zeckendorf representation.

We claim that the Zeckendorf representation of a does not contain f_{j-1} or f_j. By definition of a, we have $a < f_{j+1} - f_j = f_{j-1}$. Since $a < f_{j-1}$, the Zeckendorf representation of a does not contain Fibonacci numbers adjacent to f_j. Therefore, we can add the Zeckendorf representation of a to f_j to obtain a Zeckendorf representation of n.

5. A. We'll give two proofs of the identity $f_n = 1 + \sum_{k=1}^{n-2} f_k$.

Proof 1. By definition of the Fibonacci sequence, we have, for $n > 2$,

$$f_n = f_{n-1} + f_{n-2}. \tag{1}$$

Similarly, $f_{n-1} = f_{n-2} + f_{n-3}$ and $f_{n-2} = f_{n-3} + f_{n-4}$. We can therefore rewrite equation (1) as:

$$f_n = f_{n-2} + f_{n-3} + f_{n-4} + f_{n-3}. \tag{2}$$

Let us focus on the repeated f_{n-3} term in equation (2). Expanding this term yields:

$$f_n = f_{n-2} + f_{n-3} + f_{n-4} + f_{n-5} + f_{n-4}. \tag{3}$$

Carrying out this same expansion with the repeated f_{n-4} term in equation (3) yields:

$$f_n = f_{n-2} + f_{n-3} + f_{n-4} + f_{n-5} + f_{n-6} + f_{n-5}. \tag{4}$$

We can iterate through this process of expanding the repeated term in the equation until we are left with:

$$f_n = f_{n-2} + f_{n-3} + \ldots + f_2 + f_1 + f_2. \tag{5}$$

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Substituting $f_1 = f_2 = 1$ into equation (5) gives us:

$$f_n = f_{n-2} + f_{n-3} + \ldots + 1 + 1 + 1. \qquad (6)$$

Note that we can rewrite equation (6) using summation notation as: $f_n = 1 + \sum_{k=1}^{n-2} f_k$, as desired.

Proof 2. Define the sequence $g_n = 1 + \sum_{k=1}^{n-2} f_k$. We use induction to show that $g_n = f_n$ for all n. Note that $g_1 = g_2 = 1$. Let $n > 1$ be a positive integer and assume $g_k = f_k$ for $k \le n$. We compute,

$$g_{n+1} = 1 + \sum_{k=1}^{n-1} f_k = (1 + \sum_{k=1}^{n-2} f_k) + f_{n-1} = g_n + f_{n-1} = f_n + f_{n-1} = f_{n+1}.$$

B. By definition of the Fibonacci sequence, we have $f_n = f_{n-1} + f_{n-2}$, for $n > 2$.

Base cases: $r_1 = r_2 = 1$.

Inductive hypothesis: Suppose the pattern works for all f_k such that $k < n$.

Inductive step: We may assume $n > 2$. We want to show that $r_n = r_{n-2} + 1$.

First, we claim that every Fibonacci partition of f_n must have either f_n or f_{n-1} as a part. To see this, recall the lemma proven in Problem 5A, namely that

$$f_n = 1 + \sum_{k=1}^{n-2} f_k .$$

This says that f_n is one more than the sum of *all* the first $n - 2$ Fibonacci numbers. A Fibonacci partition, by definition, cannot include a Fibonacci number more than once (with the exception of 1, which can occur twice since $f_1 = f_2 = 1$). Since any sum of Fibonacci numbers that are a subset of the first $n - 2$ Fibonacci numbers will be less than or equal to the sum of all of them, we conclude that there is no Fibonacci partition of f_n that does not include either f_n or f_{n-1}.

Now observe that the only Fibonacci partition of f_n that has f_n as a part is (f_n). Hence, all other Fibonacci partitions of f_n have f_{n-1} as a part. By the Fibonacci recursion relation, we know that $f_n = f_{n-1} + f_{n-2}$. This shows that in a Fibonacci partition of f_n that has f_{n-1} as a part, the parts other than f_{n-1} must form a Fibonacci partition of f_{n-2}. Conversely, f_{n-1} can be added to any Fibonacci partition of f_{n-2} to obtain a Fibonacci partition of f_n. Therefore, there are r_{n-2} Fibonacci partitions of f_n that have f_{n-1} as a part.

Tallying up these Fibonacci partitions, we conclude that $r_n = r_{n-2} + 1$, as desired.

Calendar

Session 29: (all dates in 2021)

September	9	Start of the twenty-ninth session!
	16	
	23	
	30	
October	7	
	14	
	21	
	28	
November	4	
	11	
	18	
	25	Thanksgiving - No meet
December	2	

Girls' Angle has been hosting Math Collaborations at schools and libraries. Math Collaborations are fun math events that can be adapted to a variety of group sizes and skill levels. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

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Key: n.pp = number n, page pp

Girls’ Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant’s Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ______________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don’t like math, what don’t you like? If you love math, what do you love? What would you like to get out of a Girls’ Angle Membership?

The $50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- ☐ Enclosed is a check for $50 for a 1-year Girls’ Angle Membership.
- ☐ I am making a tax free donation.

Please make check payable to: **Girls’ Angle**. Mail to: Girls’ Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

Girls' Angle

A Math Club for Girls

Girls' Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is $20/meet for members and $30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle is based in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, founder and director of the Exploratory
Yaim Cooper, Institute for Advanced Study
Julia Elisenda Grigsby, professor of mathematics, Boston College
Kay Kirkpatrick, associate professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, assistant dean and director teaching & learning, Stanford University
Lauren McGough, postdoctoral fellow, University of Chicago
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, associate professor, University of Utah School of Medicine
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Liz Simon, graduate student, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, associate professor, University of Washington
Karen Willcox, Director, Oden Institute for Computational Engineering and Sciences, UT Austin
Lauren Williams, professor of mathematics, Harvard University

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls’ Angle: Club Enrollment Form

Applicant’s Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you’d like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter’s name in any way. Do we have permission to use your daughter’s image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls’ Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls’ Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: _______________________

(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is $216 for one session (12 meets)	☐ I will pay on a per meet basis at $30/meet.
☐ I will pay on a per meet basis at $20/meet.	☐ I’m including $50 to become a member, and I have selected an item from the left.

☐ I am making a tax free donation.

Please make check payable to: **Girls’ Angle**. Mail to: Girls’ Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: ___________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____