## Girlsf Bulletin <br> December 2012 • Volume 6 • Number 2

To Foster and Nurture Girls' Interest in Mathematics


## From the Founder

Math is an art - a fact easily missed, because so much of school math is about learning what others created centuries ago. Yet, fascinating unresolved questions inspire mathematicians to continually conjure up novel conceptions with novel properties. If you think all math problems are cut and dry, try the problems on page 21. Even though each is a math problem, two solvers are likely to produce completely different, but valid, solutions.

- Ken Fan, President and Founder


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## Girls’ Angle Bulletin

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Girls’ Angle welcomes submissions that pertain to mathematics.

Editor: Jennifer Silva
Executive Editor: C. Kenneth Fan

## Girls’ Angle:

A Math Club for Girls
The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

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On the cover: Red Sea Pearls by Radmila Sazdanović. See this issue's interview. Printed with permission from Radmila Sazdanović.

## An Interview with Radmila Sazdanović, Part 1

Radmila Sazdanović is a postdoctoral fellow in mathematics at the University of Pennsylvania. She is a native of Serbia.

Ken: Hi Radmila, I'd like to start with beginnings. How and when did you become interested in mathematics?

Radmila: I am not sure. It seems to me that I have always been interested in mathematics - but my perception of what mathematics is has evolved over time. I had many wonderful math teachers, but it was my geometry teacher in 9th grade whom I have to thank for conveying the idea that math is much more than solving school or competition problems. He helped me see that math is about reasoning and posing new problems.

Ken: What was the first mathematical idea that got you excited about math?
Radmila: It happened in that geometry class in 9th grade. We had learned that a line is determined by two points. One of the homework problems was to show (from the axioms) that there exists a third point on the line!!! I did not know where to start or how to think about it, but eventually I managed to solve it. But then it got even more interesting! We learned about other kinds of geometries, hyperbolic and elliptical, which are contradictory (to each other and the Euclidean geometry) but still exist - even co-exist!

Ken: Could you please tell us about some of the highlights of your journey to becoming a mathematician?

Radmila: When I was in middle school, math was all about competitions. I was selected because I was good at math; it was a natural thing to do. But because of that I was exposed to problems which were much different - and much more fun - than the ones I had experienced in school. That made me like math even more! I also really enjoyed traveling and being with kids who shared my interests. Now, I am happy to be a part of the math community, and grateful to have wonderful colleagues and collaborators and great memories from numerous conferences and summer schools.

Ken: What are some of the key qualities that you possess that enabled you to succeed in mathematics? Did math always come easily to you?

Radmila: Math demystified: you have to work hard. That is it. Even if you are very talented you will still have to be persistent. That is also part of the beauty. Understanding a new concept or solving a problem is more satisfying for having been a challenge.

Ken: Some of your papers have the word "categorification" in the title. Our readers have probably heard the words "category" and "categorize," but not categorification. What is it?

Radmila: I have been asked that question by many distinguished mathematicians. I am not sure I have a satisfactory answer.

When I was a kid and my cousins came to visit, my mom would give us candy. When we divided it up, we did so by producing an explicit bijection between the piles: one for me, one for
you, one for me, one for you, etc. Eventually some clever one of us discovered that we could just count them. But if not all the candy is the same, counting and dividing might result in me having all the hard candies, and my cousin having all the chocolate. That just won't do. Counting and dividing "decategorifies" the candy distribution, by forgetting differences by adding different kinds of things together. Categorification is the project of discovering which

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Thank you and best wishes,
Ken Fan
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Ken: In your WIM video, you explained a result that you found interesting when you were a student. Here, would you explain to us a favorite result that you proved?
(content removed from Electronic version - available in print version)

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Ken: How tantalizing! Do you have any advice for how to best approach learning mathematics?
To be continued...
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Girls,

# The Stable Marriage Problem, Part $2^{1}$ 



Storyboard by Larry Guth, Grace
Lyo, Amy Pasternak, Elizabeth Lyo, Amy Pasternak, Elizabeth


TABLE 1. The girls' preferences


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## Cones in Your World

by Taotao Liu I edited by Jennifer Silva
Even now, with Christmas over, Christmas trees are still everywhere I look: in the living room, in malls, in plazas outdoors ... As I walk past them, I can't help but think about cones. Come to think of it, cones (or things that almost look like cones) are all over the place. Kitchens have funnels, roads have pylons, cars emit conical light beams, ice cream parlors hand out cones you can eat, cheerleaders shout through cone-shaped megaphones, and party-goers wear cones on their heads, like Gandalf. How many other places can you think of where there are cone-shaped things?
 How about ... mathematics?

Let's take a look at cones from a mathematician's point of view. In this article, we'll focus on the basic cone that has a circular base and a vertex located directly above the center of that base. In other words, if you were to draw the line connecting the vertex of the cone to the center of its base, that line would form a right angle with the base. This kind of cone is technically known as a right circular cone. There is a more general notion of cone that allows the vertex to be off-center (a skew cone) and the base to be something other than a circle. In the most general definition, pyramids are cones. But here, we'll focus on right circular cones.

## Construction

Try to visualize the following construction in your mind's eye before you execute it with real paper. By trying to imagine first, then making a concrete model, you will be developing a useful thinking skill.

Take a sheet of paper. You're going to roll it up in a special way, as if you were going to make a scroll (like the way posters are often packaged). However, instead of allowing the corners of the paper to line up with the edges, tilt one of the corners toward the inside of the scroll as you're rolling it. Keep tilting it until that end of the scroll is closed off into a point. Continue rolling up the rest of the paper around the surface of the cone that is being formed, while maintaining the pointed end.

If you're just using any old piece of paper, the edge of the cone won't be smooth like it would be on a megaphone. To make a smooth circular edge, you can cut the excess paper away. Use glue or tape to hold your cone together.

## Surface Area

Suppose we had to make cones using the method described above for a specific purpose, for example, to make party hats. It would be nice if there were some way to determine how much material we would need for each cone so that we could figure out how much material to

purchase to make the hats. Since the surface of a cone is two-dimensional, what we need to know is its surface area. What is the surface area of a right circular cone?

I've actually never bothered to memorize the formula for the surface area of a cone. It's a fun formula to derive because each time I derive it, I'm reminded of some of its beautiful properties. First of all, the cone consists of a flat part, the circular base, and the side part, which is its lateral surface. We won't worry about the area of the circular base since party hats don't have that part. (Though if you did have to account for it, how would you determine its contribution to the total surface area of the cone?) So let's think about the

## lateral surface area.

The lateral surface is a rounded surface. How can we compute its area? The key is that even though the lateral surface is rounded, it can be built by rolling up a flat piece of paper.
Let's cut along a line extending from the vertex straight down to a point on the perimeter of the base, then unroll the cone back to a flat piece.

What shape is this unfurled cone?
It is, in fact, a sector of a circle! To see this, consider a point on the perimeter of the base of the cone. The straight line segment connecting this point to the apex sits inside the lateral surface. When the lateral surface is unfurled, the line segments connecting points on the perimeter of the base to the apex become a bunch of radial lines fanning out from what used to be the apex of the cone. Every point on the lateral surface is on one of these radial lines. Because a right circular cone has 360degree rotational symmetry, all of these radial lines have the same length, known as the lateral height of the cone. We can therefore see that the unfurled lateral surface is a sector of a circle whose radius is the lateral height of the cone.

What fraction of the circle is the resulting sector?


We can find out by traveling along the circumference of the circle. As we travel, we pay attention to what fraction of the journey runs along what used to be the perimeter of the base of the cone.

Let's give some names to key lengths in this story so that we can write down a formula for the cone's lateral surface area. Let $r$ be the radius of the circular base of the cone. Let $l$ be the lateral height. It's often useful to label the height of a cone, too. The height of a cone is the distance between the apex and the base. But we don't need the height in our formula for the lateral surface area (although note that $r, l$, and the height of the cone are related by the Pythagorean theorem).

The part of the circle of radius $l$ that the unfurled cone occupies is the ratio of the circumference of the base of the cone to the circumference of the circle of radius $l$. Because all circles are similar to each other, the ratios of corresponding lengths will all be equal. So the ratio
of the circumference of a circle of radius $r$ to that of a circle of radius $l$ is equal to the ratio of their radii, which is $r / l$. To find the lateral surface area of our cone, then, all we have to do is multiply the area of a circle of radius $l$ by the fraction $r / l$. Since the area of a circle with radius $l$ is equal to $\pi l^{2}$, the lateral surface area of the cone is $\left(\pi l^{2}\right) r / l=\pi l r$.

## Flatness of a Cone

A key feature of cones that we used to get a formula for the lateral surface area is that the lateral surface can be flattened. The ability to flatten a surface completely without distortion is a very special property. For example, you cannot flatten the surface of a sphere. That's why maps of the globe are a whole subject of intrigue. If the earth were cone-shaped, mapmakers would have a much easier job.

The "flatten-out-ability" of a cone also enables us to determine the shortest distance between any two points on the lateral surface of a cone while staying within this surface. That is, if you lived on a cone-shaped planet, how could you compute the shortest distance between two of its cities? All you would have to do is unfurl the cone, then measure the distance between the two cities with a ruler! One caveat: if the two points of interest happen to be near to and on opposite sides of the slit made for unfurling, and you blindly forge ahead with this measuring technique, you'll get the wrong answer for the shortest distance. Why? What's the solution?

## Volume

What about the volume of a cone? See page 15 .

## Take it to Your World

Suppose you are organizing a birthday party for your five-year-old brother. He wants to make 12 cone-shaped party hats out of some funky holographic paper, and he'd like the hats to be 24 cm in height with a circular base 14 cm in diameter. If the holographic paper costs $\$ 40$ per square meter, what's the minimum amount of money you would have to spend on holographic paper if you could purchase exactly the amount of paper needed to make these hats? (Send your answers to girlsangle@gmail.com.)

Go into a dark room with a flashlight. Study the various light shapes you can make by shining the flashlight on the wall. These shapes are cross-sections of the cone, and their outlines are known as conic sections.

We saw that with a few cuts, a cone can be unfurled into a flat shape. Not all solids have this property of "flatten-out-ability": however, when one does, the unfolded, flat object is called a net of the solid. What kinds of nets can you obtain by flattening out a cube? A square pyramid? An octahedron? A cylinder? Can you prove that it is impossible to flatten out a sphere without introducing distortion? If a curved surface can be slit in places and then unfolded to lie on a flat surface, we say that it has zero curvature. Look up the curvature of a sphere. Are you surprised?

Suppose a world is shaped like a cone, and its inhabitants live on its surface. There are cities scattered on its lateral surface and on its circular base. Earlier, we explained how you can determine the distance inhabitants would have to travel between cities located on the lateral surface of the cylinder (assuming that they must walk along the surface of the cylinder). How can you compute the distance between one city on the lateral surface and any city on the circular base?

## Notation

by Ken Fan
Through time, ways to notate mathematical ideas have become standard. Learning the notational conventions is important for communication. The best way to learn new notation is the same way we learned the alphabet: repetition. Practice makes perfect!

## Double Factorials

In her interview, Dr. Sazdanović made use of the "double factorial" notation: $n$ !!. This notation does not mean to apply the factorial twice. Instead, $n!$ ! is the product $n(n-2)(n-4) \cdots$, where the product goes down by twos until you reach either 1 or 2 .

Here's a table of the first few factorials and double factorials:

| $\boldsymbol{n}$ | $\mathbf{1}$ | $\mathbf{2}$ | $\mathbf{3}$ | $\mathbf{4}$ | $\mathbf{5}$ | $\mathbf{6}$ | $\mathbf{7}$ | $\mathbf{8}$ | $\mathbf{9}$ | $\mathbf{1 0}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\boldsymbol{n}!$ | 1 | 2 | 6 | 24 | 120 | 720 | 5040 | 40320 | 362880 | 3628800 |
| $\boldsymbol{n}!!$ | 1 | 2 | 3 | 8 | 15 | 48 | 105 | 384 | 945 | 3840 |

Notice that $n!=n!!(n-1)!!$. Also, $(2 n-1)!!=\frac{(2 n)!}{2^{n} n!}$ and $(2 n)!!=2^{n} n!$.
Because the double factorial can be expressed in terms of the factorial, you might wonder why it is used at all. Like much notation, it starts as a matter of convenience, and sometimes takes on a life of its own. Here are 4 things that double factorials conveniently count. Can you convince yourself of the validity of each one?

1. The number of ways to pair up $2 n$ points so that each point belongs to exactly one pair is equal to $(2 n-1)$ !!. (This was Dr. Sazdanović's application.) Another way to interpret this fact is that $(2 n-1)!!$ is the number of ways you can split $2 n$ tennis players into $n$ doubles pairings.
2. The number of symmetries of the $n$-dimensional hypercube is ( $2 n$ )!! (including reflections). For example, a square has $4!!=8$ symmetries: the 4 rotations in $0,90,180$, and 270 degrees, the 2 reflections over its diagonals, and the 2 reflections over the 2 lines that bisect the square and are square with the sides. A cube has $6!!=48$ symmetries. Can you find them all?
3. These numbers even related to marriages: $2 n$ people from $n$ married couples line up single file to enter a movie theater. How many ways can they be ordered so that married people are adjacent? The answer is ( $2 n$ )!!.
4. A store window decorator wants to display $n$ different creatures that are made out of chocolate. She has a white chocolate and regular chocolate version of each creature, but she will only show one version of each creature in the window display. How many different ways can she arrange the $n$ creatures in a row? The answer is ( $2 n$ )!!.

## Cone Volume

by Ken Fan
Taotao discusses right circular cones in this issue's Math In Your World. Here, we show that the volume of such a cone is precisely $B h / 3$, where $B$ is the area of the base of the cone and $h$ is its height. That is, if you hone a cylinder to a pointy end, you lose two-thirds of its original volume.

## Calculus



Calculus is the subject that is perfectly suited to computing volumes such as these. If you know calculus, then you know that this problem can be resolved by computing a simple integral. If you don't know calculus, you've got something big to look forward to! We'll proceed without calculus, but we will have to use some facts that can only be proven rigorously with calculus or with tools that are closely related to calculus.

## Building Blocks

The triangle is the 2-dimensional analog of the cone. One way to find the area of a triangle, or any polygon, is to cut it into pieces that can be rearranged to form a rectangle. Unfortunately, in 3 dimensions, it is impossible to cut up a cone, or even the generic square pyramid, into a finite number of pieces that can be rearranged to form a block shape. ${ }^{1}$ Even so, we will find the volume of the cone by comparing it to the volume of a block. We will do this by building a chain that connects the right circular cone with a cube. Each link in the chain will connect one solid to another in a simple enough way that their volume relationship is well understood. So many mathematical arguments are just like such chains!


## Step 1: Fit the cone inside a square pyramid.

First, circumscribe a square around the base of the cone. Form the square pyramid with this square as its base and with the apex that is the same point as the apex of the cone. This square pyramid snugly contains the cone. The surfaces intersect along four lateral lines and the entire base of the cone.

Notice that every plane parallel to the base that intersects this structure intersects the cone in a circle, and it intersects the square pyramid in the square that circumscribes the circle. Because all squares and circles are similar to each other, the ratio of the area of a circle to its circumscribed square is a fixed constant that does not depend on size. If the ratios of the areas of the cross-sections of the cone and square pyramid are constant for every planar cross-section parallel to the base, then the ratio of the volumes of the cone and square pyramid will be equal to this constant. This is one of the inferences that requires some kind of calculus machinery to prove rigorously; we'll return to this point later. Here, this ratio happens to be $\pi / 4$.

[^0]
## Step 2: Apply Cavalieri's principle.

Cavalieri was an Italian mathematician who lived in the first half of the $17^{\text {th }}$ century. Let's imagine two solids, and we'll see Cavalieri's legacy at work. Suppose you can orient and position the solids side by side so that every horizontal plane intersects them in equal-area crosssections. Cavalieri observed that the two solids must have the same volume (hence the subsequently-named Cavalieri's principle, which explains this observation). This is another fact that requires calculus or something akin to it to prove rigorously.

We can apply
Cavalieri's principle to see that the square pyramid from Step 1 has the same volume as any pyramid with the same base and height. In other words, we can move the apex within a plane parallel to the base and obtain a family of pyramids of equal volume. So if we could figure out the volume of any one of
 these square pyramids, we would be done.

Let's focus our attention on the particular square pyramid where the apex is moved directly above one of the corners of the square base. We'll call this particular square pyramid $p$.


## Step 3: Stretch (or compress) $p$.

Next, uniformly stretch or compress $p$ only in the vertical direction so that its height is the same length as the sides of its base, and call the resulting square pyramid $P$. Notice that $P$ sits snugly inside a cube.

## Step 4: Use 3 copies of $\boldsymbol{P}$ to make a cube.

If you take 3 copies of $P$, you can fit them together to form a cube. Do you see how? (If you're having trouble visualizing this, make models of 3 congruent copies of $P$ and figure out how to fit them together to make a perfect cube.)

At last, we are able to compare the volume of our cone to that of a cube.
Exercise: Unravel these 4 steps and conclude that the volume of a cone is $B h / 3$, where $B$ is the area of its circular base and $h$ is its height.

## Generalization

Taotao mentioned a more general definition of cone, under which pyramids are considered to be examples of cones. The more general definition of cone goes like this: Fix a region in a plane. It does not have to be a circle. You can use a square, rectangle, triangle, ellipse, or any shape you wish, even nameless shapes. Pick a point outside of the plane. This point will be the apex of our cone. To build the cone, take the volume swept out by every line segment that connects the apex to a point of the base.

Many solids can be constructed in this manner. The right circular cones that Taotao studied are the special case where the base is a circular disc and the apex is located directly above the center of this disc. All pyramids are cones of the more general type, from regular tetrahedra to the pyramids of Egypt.

We are mentioning this more general notion of cone because each of the 4 steps above that was applied to a right circular cone can equally well be applied to any of these more general cones. In Step 1, you can use any square that contains the base; it does not have to be a circumscribing square. And once you get by Step 1, the rest of the steps are exactly the same as before because none of the subsequent steps refer directly to the original cone.

Thus, the volume formula $B h / 3$, where $B$ is the area of the base and $h$ is the height, is valid for every single one of these generalized cones, too!

## The volume of a sphere of radius $R$ is $\pi^{2} R^{3} / 2 ? ? ?$

Let's return to the inference made in Step 1. It is a subtle one. It may seem intuitively clear that the ratio of volumes is the same as the ratio of the areas of cross-sections if the area ratios are constant for an entire family of parallel planes, but be careful about accepting this conclusion without some careful thought.

One application of this observation is that the volume of a cylinder with height $h$ and base radius $r$ should be $\pi / 4$ times that of a square prism of the same height and with a square base that has side length $2 r$. That's because if you slip the cylinder into the square prism, every crosssection with a plane parallel to the base is the same: a circle inscribed in a square. Indeed, such a square prism has volume $h(2 r)^{2}=4 h r^{2}$, and such a cylinder has volume $(\pi / 4)\left(4 h r^{2}\right)=\pi h r^{2}$.

Now consider a sphere of radius $R$. It sits snugly inside of a cylinder with base radius $R$ and height $2 R$. Think of the planes that contain the axis of the cylinder. You can pick one of them and get the others by rotating it around the axis.

Notice that every one of these planes intersects the sphere in a familiar picture: a circle inscribed in a square, just like the cone and the cylinder! (Note that these planes are not parallel to the base of the cylinder; they are perpendicular to it.)

Since the ratio of the areas of the cross sections is constant for a whole family of planes that sweep out the full spherical solid, then by the same reasoning used in Step 1, shouldn't we conclude that the ratio of the volume of the sphere to that of the cylinder is equal to the ratio of the area of a circle to that of its circumscribing square?

The ratio of the area of a circle to the circumscribing square is $\pi / 4$. The volume of a cylinder with base radius $R$ and height $2 R$ is equal to $2 \pi R^{3}$. Therefore, the volume of a sphere of radius $R$ is $(\pi / 4)\left(2 \pi R^{3}\right)=\pi^{2} R^{3} / 2$. Right ?

Huh? Well, we know that isn't really the volume of such a sphere. The correct formula for the volume of a sphere in terms of its radius $R$ is $4 \pi R^{3} / 3$. So what went wrong? Why doesn't the idea of Step 1 work this time?


# Fermat's Little Theorem, ${ }^{1}$ Part 2 

Robert Donley runs the YouTube channel MathDoctorBob, which has over 650 videos on close to 20 math subjects.

by Robert Donley

Last time, we introduced Fermat's Little Theorem: If we choose any prime $p$ and any integer $m$, then $p$ divides $m^{p}-m$. For instance, with $m=5$ and $p=13$, the theorem says that 13 divides evenly into $5^{13}-5$. Stated using the notation we introduced for modular arithmetic, this becomes $m^{p}=m(\bmod p)$. (Recall that we write $a=b(\bmod n)$ if and only if $n$ divides evenly into $a-b$.)

We gave a proof by induction that used the binomial theorem and a nice observation about the prime-numbered rows of Pascal's triangle. But there are other proofs that reveal deeper insights, and we'll explore another in this article.

Fermat's little theorem is about the remainder you get when you divide $m^{p}$ by $p$. This suggests exploring the remainders of other powers of $m$ upon division by $p$. For example, consider the sequence of powers of 3 :

$$
3,9,27,81,243,729,2187,6561,19683,59049,177147,531441, \ldots
$$

What happens if we reduce this sequence modulo 7? That is, what do we get if we look at the remainders of these numbers after we divide by 7? We get this:

$$
3,2,6,4,5,1,3,2,6,4,5,1, \ldots
$$

We urge you to compute the above sequence by hand. Note that computing the sequence of powers of 3 modulo 7 does not actually require computing powers of 3 . That's because the remainder of a product is the product of the remainders. If $a=b(\bmod n)$ and $c=d(\bmod n)$, then $a c=b d(\bmod n)$. So to compute the remainders of powers of 3 , modulo 7, we never have to compute the original sequence of powers of 3 . Instead, we can proceed like this:


In this way, we can compute the sequence of remainders, modulo 7 , of powers of $0,1,2,3,4,5$, and 6 rather quickly. Help us complete the following table:

| Table of Powers Modulo 7 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\boldsymbol{n}$ | $\mathbf{1}$ | $\mathbf{2}$ | $\mathbf{3}$ | $\mathbf{4}$ | $\mathbf{5}$ | $\mathbf{6}$ | $\mathbf{7}$ | $\mathbf{8}$ | $\mathbf{9}$ | $\mathbf{1 0}$ | $\mathbf{1 1}$ |  |  |  |  |  |  |  |
| $\mathbf{0}^{\boldsymbol{n}}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |  |  |  |  |  |  |  |
| $\mathbf{1}^{\boldsymbol{n}}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |  |  |  |  |  |  |  |
| $\mathbf{2}^{\boldsymbol{n}}$ | 2 | 4 | 1 | 2 | 4 | 1 | 2 | 4 | 1 | 2 | 4 |  |  |  |  |  |  |  |
| $\mathbf{3}^{\boldsymbol{n}}$ | 3 | 2 | 6 | 4 | 5 | 1 | 3 | 2 | 6 | 4 | 5 |  |  |  |  |  |  |  |
| $\mathbf{4}^{\boldsymbol{n}}$ | 4 | 2 | 1 | 4 | 2 | 1 | 4 | 2 | 1 | 4 | 2 |  |  |  |  |  |  |  |
| $\mathbf{5}^{\boldsymbol{n}}$ | 5 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| $\mathbf{6}^{\boldsymbol{n}}$ | 6 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |

[^1]What patterns do you see? Which patterns persist for prime numbers other than 7? Also, why did we stop the rows at powers of 6 and not list powers of $7,8,9$, etc.?

Notice that 0 only occurs in the sequence of powers of 0 . In other words, if an integer is divisible by 7, then so are its (positive) powers, and vice versa. This holds for all primes: an integer is divisible by $p$ if and only if any one of its positive powers is divisible by $p$.

Do you also see that each sequence is periodic? Let's try to understand this periodicity with respect to any prime number $p$.

First, notice that there are only a finite number of remainders, modulo $p$, namely $0,1,2$, $3, \ldots, p-1$. Since there are only finitely many different remainders, the sequence must reuse remainders. So there must be some $a$ and $b$, with $a<b$, where $m^{a}$ and $m^{b}$ have the same remainder, modulo $p$. Then $p$ must divide their difference: $m^{b}-m^{a}=m^{a}\left(m^{b-a}-1\right)$. When a prime divides a product, it must divide one or both factors. So either $p$ divides $m^{a}$ or $p$ divides $m^{b-a}-1$. (Note that, in this case, $p$ can only divide one of the factors.) If $p$ divides $m^{a}$, then $p$ must divide $m$, and we are in the row that's filled with zeros. If $p$ does not divide $m$, then $p$ must divide $m^{b-a}-1$, that is $m^{b-a}=1(\bmod p)$. If we multiply both sides by $m$, we find $m^{b-a+1}=m(\bmod p)$. This tells us that the sequence must, in fact, return to its first value. So, not only is every sequence periodic, but the periodic pattern begins right away. (It's not like, for instance, the digits in the decimal expansion of $13 / 60=0.21666 \ldots$, where the decimal digits are eventually periodic, but the repeating pattern doesn't begin right away.)

So, just by looking at sequences of powers, we have discovered something very close to Fermat's little theorem. We've found that for any $m$, there exists some integer $q>1$, such that $m^{q}=m(\bmod p)$. To recover Fermat's little theorem, we have to show that $q$ can be taken to be $p$ itself.

The situation where $p$ divides $m$ produces a sequence of zero remainders, so let's put this case aside and concentrate on the cases where $p$ does not divide $m$.

When $p$ does not divide $m$, each sequence begins with $m$, then goes through some of the remainders, before eventually returning to $m$. The way we deduced this, we also saw that there will be some power of $m$ that is equal to 1 modulo $p$. Let $r$ be the smallest positive integer such that $m^{r}=1(\bmod p)$.

The remainders of $m, m^{2}, m^{3}, \ldots$, and $m^{r}$ must all be different, because if two of them were the same, we could argue as before that there is a smaller power of $m$ equal to 1 modulo $p$. This tells us that $r<p$. Furthermore, since $m^{r}=1(\bmod p)$, the sequence starts repeating with $m^{r+1}=m(\bmod p)$. So we deduce that the sequence $m^{n}$, modulo $p$, is periodic right away with period $r$, where $r$ is the smallest positive integer such that $m^{r}=1(\bmod p)$.

Let $x$ be a number not divisible by $p$. Denote by $R_{x}$ the set of remainders modulo $p$ of the numbers $x m, x m^{2}, x m^{3}, \ldots, x m^{r}$. Notice that no two of these numbers leave the same remainder because if $x m^{k}=x m^{j}(\bmod p)$, with $j<k$, then $p$ divides $x m^{j}\left(m^{k-j}-1\right)$. But since $p$ doesn't divide $x$ or $m$, we would find that $p$ divides $m^{k-j}-1$, which contradicts the minimality of $r$. Thus, $R_{x}$ contains exactly $r$ elements.

Now fix two numbers $x$ and $y$, neither of which is divisible by $p$. We claim that either $R_{x}$ and $R_{y}$ are equal to each other, or they are completely disjoint. (That is, they cannot have nonempty intersection without being equal to each other.) For suppose that there is an element in both. Then we would have

$$
x m^{k}=y m^{j}(\bmod p)
$$

for some $0<k, j \leq r$. Multiply both sides by $m^{r-k}$ to obtain:

$$
x m^{r}=y m^{j+r-k}(\bmod p)
$$

Since $m^{r}=1(\bmod p)$, this means that $x=y m^{j+r-k}(\bmod p)$. Now, $R_{y}$ is the set of remainders, modulo $p$, of the numbers $y m, y m^{2}, y m^{3}, \ldots, y m^{r}$. Since the sequence $m^{n}$ is periodic with period $r$, this is the same as the remainders, modulo $p$, of the numbers $y m^{s}, y m^{s+1}, y m^{s+2}, \ldots, y m^{s+r-1}$, for any nonnegative integer $s$. In particular, if you take $s=j+r-k$, we see that $R_{y}$ is the set of remainders, modulo $p$, of the numbers $y m^{j+r-k}, y m^{j+r-k+1}, y m^{j+r-k+2}, \ldots y m^{j+r-k+r-1}$. And since $x=y m^{j+r-k}(\bmod p)$, this is the set of remainders, modulo $p$, of $x m, x m^{2}, x m^{3}, \ldots, x m^{r}$, which is just $R_{x}$. So $R_{y}=R_{x}$.

Since every remainder from 1 to $p-1$ is contained in some $R_{x}$ (after all, if $1 \leq y \leq r$, then $y$ is in $R_{y}$ ), we see that $\{1,2,3, \ldots, p-1\}$ is the union of the $R_{x}$ for $x=1,2,3, \ldots, p-1$. And because $R_{x}$ and $R_{y}$ are either disjoint or equal, and all the $R_{x}$ have $r$ elements in them, we conclude that $\{1,2,3, \ldots, p-1\}$ can be expressed as a disjoint union of subsets each of size $r$. That is, $r$ must divide evenly into $p-1$. If we define $r^{\prime}$ by $r r^{\prime}=p-1$, then we find that

$$
m^{p-1}=m^{r r^{\prime}}=\left(m^{r}\right)^{r^{\prime}}=1^{r^{\prime}}=1(\bmod p) .
$$

Multiplying both sides by $m$, we find that $m^{p}=m(\bmod p)$. That's exactly the statement of Fermat's little theorem!

In the proof we just gave, it was useful to look at the smallest positive integer $r$ such that $m^{r}=1(\bmod p)$. This number is called the order of $m$, modulo $p$ and is often written $o_{p}(m)$, or just $o(m)$ if it is clear from the context what the modulus is.

To review, we have shown that if $p$ does not divide $m$, then $o(m)$ exists and divides $p-1$. We've seen that the sequence of remainders of powers of $m$, modulo $p$, is right away periodic with period $o(m)$. We can make a more precise version of Fermat's little theorem by saying that $m^{n}=m(\bmod p)$ precisely when $n=k o(m)+1$ for nonnegative integers $k$. And we've seen that if $p$ does divide $m$, then the remainder of any power of $m$, when divided by $p$, is 0 .

Challenge: Can you show that for any prime number $p$, there is an $m$ such that $o(m)=p-1$ ?
Another consequence of our observations is that whenever $p$ does not divide $m$, there exists an $m^{\prime}$ such that $m m^{\prime}=1(\bmod p)$. We can just take $m^{\prime}=m^{o(m)-1}$. In other words, when we work modulo $p$ where $p$ is a prime number, all numbers not equal to 0 modulo $p$ are invertible. (For another proof, see page 27.)

This means that the equation $a x=b(\bmod p)$ is always solvable when $a \neq 0(\bmod p)$. For example, since $2 \cdot 4=1(\bmod 7)$, we can find the solution to $4 x=3(\bmod 7)$ by multiplying both sides by 2 :

$$
\begin{aligned}
4 x & =3(\bmod 7) \\
2 \cdot 4 x & =2 \cdot 3(\bmod 7) \\
x & =6(\bmod 7)
\end{aligned}
$$

Contrast this with the situation when we work modulo 6 , which is composite. What is the solution to $4 x=3(\bmod 6)$ ? In fact, there is no solution! After all, if $4 x=3(\bmod 6)$, then 6 must divide evenly into $4 x-3$. That is, there would exist some integer $k$ such that $6 k=4 x-3$. But the left-hand side, $6 k$, is even, whereas the right-hand side, $4 x-3$, is always odd, so such an equation is impossible.

Notice that the remainders upon division by 6 of the powers of 4 are always equal to 4 . You never get 1. Check that the only remainders modulo 6 that are invertible are 1 and 5. What can you say about power sequences modulo $n$ when $n$ is composite?

## Invent!

by Ken Fan

The following tasks ask you to make something. Each task has many solutions, so be creative! We welcome you to send your solutions to us at girlsangle@ gmail.com.

Invention 1. You and your friends order a pizza at a pizza parlor. It happens that you form a party of 7 . Invent a way to divvy up a pizza into 7 equal parts.

Invention 2. Take a look at US flags through the ages. There weren't always 50 States. Now imagine that a $51^{\text {st }}$ State is annexed. Design a new US flag for that scenario.


Invention 3. The Pythagorean theorem has been proven in dozens of different ways, including one invented by former US President James Garfield. Invent your own proof of the Pythagorean theorem.

Invention 4. Fonts can be created to express style, BAODHASS, simplicity, and
STRENGTH. Design a special "tessellating" font. That is, design a font where every character can be used to tessellate the plane.

Invention 5. You went to a lot of trouble to bake a cake in the shape of a square prism for your a client. But when you presented the cake to the client, he balked. "I didn't ask you to bake a square cake! I asked you to bake a cake and make square servings!" Not wanting to waste all your work or the cake, figure out a way to cut this cake into pieces and reassemble them to make 2 square pieces of the same size (and each with the same height as the original cake). How about 3 square pieces? Can you figure out a way to do this for any number $N$ of pieces? (Another way to describe this task: Fix $N$. Find a way to cut a square into pieces that can be rearranged to form $N$ squares all the same size.)

Invention 6. Create an explicit formula for a function that maps the set of integers to itself and has the following additional properties:
A. Infinitely many integers are mapped to zero. B. Every integer is mapped to.

Invention 7. Mathematics can be as creative as drawing. Create a function $f(x, y)$ of two real variables such that the set of points $(x, y)$ where $f(x, y)=0$ looks like two loops (not necessarily the ones pictured at right).

Invention 8. Design a 3-dimensional solid that has a hole in it
 that can be used to tessellate space.

## COACH BARBIS CORNER

by Barbara Remmers I edited by Jennifer Silva

## Owning it: Fraction Satisfaction, Part 8

Oh dear. Here comes $3 / 7$, and not only is she looking bleary-eyed and green around the gills, but she is also towing along a boy by the necktie of his school uniform!
$\frac{\mathbf{3}}{\mathbf{7}}$ : Just who I was looking for! I need you.
You: Sure. How can I help?
$\frac{\mathbf{3}}{\mathbf{7}}$ : Explain to this lad here - Achoo! Excuse me; I have the flu - how to determine average
You: No problem. I just helped my friend with that.
$\frac{\mathbf{3}}{\mathbf{7}}$ : I knew I could count on you, sweetheart. I'm afraid my normal good cheer and infinite patience have abandoned me while I'm sick.

You: You just sit down and have some tea. I can take care of this.
$\frac{\mathbf{3}}{\mathbf{7}}$ : Thank you, darling. Also, between you and me, Igor - that's the fellow's name - was so insufferable that I feared for my survival, given my delicate state.

You: No worries. You rest .... Hey, Igor, you want to talk about average speeds?
Boy: Yes. I just don't get one of my homework problems. It's: If you drive 20 miles to the store at 10 mph and drive 20 miles home at 20 mph , then what is your average speed on your round trip? I was trying to talk to $3 / 7$ about it since she gives help sessions at my school, but I just ended up annoying her. By the way, my name's not Igor - it's Ian.

You: Don't sweat it, Ian. Let's just figure out the problem and then everyone will be happy.
Boy: Okay. So this is what's driving me nuts: If half of the trip is at 10 mph and half is at 20 mph , then how come the average speed isn't 15 mph ?

You: Well, 15 mph is the average of the two different speeds, but it isn't the average speed of the trip.

Boy: Huh? See, this is what I hate about math. It's a bunch of unrelated stuff that's full of tricks.

You: Oooh. Don't let $3 / 7$ hear you say that. She's sort of touchy.
Boy: She already did hear me. That's when she grabbed my tie and started marching over here.

You: Ah, I see. Well, anyway, the average speed on a trip actually is similar to other averages you compute.

Boy: How's that?
You: Say you have 5 test scores. Also consider that they add up to a certain total.
Boy: Okay. I don't see where you're going, but okay.
You: The average of the 5 scores is the one score that, if you had it on each and every test, would also result in the same certain total.

Boy: Okay. I don't usually think of it that way, but I see.
You: Similarly, the average trip speed is going to be the one speed that would result in the same trip taking the same amount of time.

Boy: Right. I know the definition of the average speed, but why can't I just average the two speeds?

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We're committed to producing quality math educational content and make every effort to provide this content to you for free.

We are also committed to surviving as a nonprofit!
For this issue, those who do not subscribe to the print version will be missing out on the rest of this interview with Prof. Sazdanović, most of Part 2 of the Stable Marriage Algorithm, and much of Coach Barb's Corner. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

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Thank you and best wishes, Ken Fan
President and Founder
Girls' Angle: A Math Club for Girls
$\frac{\mathbf{3}}{\mathbf{7}}$ : Honey, darling that you are, you could never replace the one and only $3 / 7$.
Boy:
You:

Boy:
$\frac{3}{7}$ :

## Hey Girls! $\quad{ }^{\text {Learn }}$ Matbematics!

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## You:

Boy:
$\frac{3}{7}$ :
You:

Boy:

You:

Boy:

## You:

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Ken Fan
President and Founder
Girls' Angle: A Math Club for Girls
You:
$\frac{3}{7}$ :
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## Notes from the Club

These notes cover some of what happened at Girls’ Angle meets. They are far from being complete. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than $5 \%$ of what happens at the club is revealed here.

Session 11 - Meet 8 Mentors: Elena Capote, Lucia Mocz, Isabel Vogt, Luyi Zhang
November 1, 2012
Here's a problem that one of our members solved: A rectangle is made by tiling unit squares. The rectangle is $n$ tiles by $m$ tiles. If the diagonal of the square is drawn in, how many tiles will be marked?

Session 11 - Meet 9 Mentors: Jordan Downey, Samantha Hagerman, Charmaine Sia, November 8, 2012 Isabel Vogt, Luyi Zhang

One theme to this meet was to construct Pascal's triangle in as many different ways as we could think of. How many different ways do you know?

Session 11 - Meet 10 Mentors: Samantha Hagerman, Isabel Vogt, Luyi Zhang November 15, 2012

The equation $a x=b$, where $a$ and $b$ are given constants is a basic one in any context where one has some kind of multiplication. In fact, it's rather amazing how rich a topic just studying this little equation in difference contexts can be!

For example, if we work within the set of integers, we are led to the notion of divisibility. We discover that 1 and -1 are the only integers whose reciprocal is also an integer. We are led to discover rational numbers.

If we work within the set of rational numbers, we are led to the concept of "closure under division" or "the existence of (multiplicative) inverses." That is, a rational number divided by a nonzero rational number will always be a rational number. Or, to put it another way, the multiplicative inverse (reciprocal) of a nonzero rational number is a rational number.

You can examine this equation in the context of complex numbers, polynomials, matrices, and groups. But at today's meet, the context was integers modulo a prime number $p$. (Also, see Robert Donley's second installment on Fermat's little theorem on page 18.)

Let $a$ be an integer. The main object was to show that if $p$ does not divide evenly into $a$, then there is an integer $x$ such that $a x=1$ (modulo $p$ ), that is, such that $p$ divides $a x-1$. One way to see this is to consider the first $p-1$ multiples of $a: a, 2 a, 3 a, 4 a, \ldots,(p-1) a$. Since $p$ does not divide $a$, nor does it divide any positive integer less than $p$, we know that $p$ does not divide any of these $p-1$ multiples of $a$. Therefore, each of these multiples of $a$ will leave a nonzero remainder if divided by $p$. There are $p-1$ different nonzero remainders and we are looking at exactly $p-1$ multiples of $a$. Could it be that these $p-1$ multiples of $a$ all leave different remainders when divide by $p$ ? That is, could these $p-1$ multiples of $a$ simply permute the remainders? If this were true, then exactly one of these multiples would leave a remainder of 1 when divided by $p$, and our result would follow.

Suppose that two of these multiples left the same remainder upon division by $p$ : say $k a$ and $j a$, with $k<j$. Then $p$ must divide $j a-k a=(j-k) a$. But we've already seen that this is not possible.

This implies that so long as $p$ does not divide $a$, the equation $a x=b$ (modulo $p$ ) always has a solution.

Can you find specific values for $a$ and $b$ so that the equation $a x=b$ does not have a solution modulo $n$ where $n$ is not a prime?

Session 11 - Meet 11 Mentors: Elenna Capote, Jordan Downey, Samantha Hagerman, November 29, 2012 Isabel Vogt, Jessica Wang

The eleventh meet of the eleventh session was most unusual because of the Cambridge blackout. The girls huddled together and did math under flashlights.

Speaking of blackouts, have you ever computed the time when power was restored by subtracting the time showing on an electric clock from the current time? Now there's a mixedbase arithmetic problem!

Special thanks to Isabel Vogt for acting as Head Mentor for this meet on very short notice.

Session 11 - Meet 12 Mentors: Jordan Downey, Jessica Wang
December 6, 2012

We held our end-of-session Math Treasure Hunt. Here are two problems from the hunt:
A square grid arrangement is made with 10 rows each containing 10 light bulbs. All the light bulbs are initially off. Person 1 flips the switches for all the light bulbs in every column. Person 1 then flips the switches for all the light bulbs in every row. Person 2 flips the switches for all the light bulbs in every second column. Person 2 then flips the switches for all the light bulbs in every second row. This continues with person $K$ first flipping the switches for all the light bulbs in every $K$ th column, then flipping the switches for all the light bulbs in every $K$ th row. After 10 people have passed through, how many light bulbs are turned on?

| 10 | 0 | 0 | 6 | $\$$ | 14 | 0 | 12 | , | $\$$ | 22 | 0 | 16 |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| $\$$ | 20 | 0 | 16 | 23 | 6 | $\$$ | 2 | 9 | 24 | 7 | $\$$ | 12 |
| 7 | 7 | 8 | $\$$ | 9 | 12 | 23 | 7 | $\$$ | 3 | 0 | $\$$ | 5 |
| 0 | 23 | 24 | 7 | $\$$ | 0 | 8 | 7 | $\$$ | 0 | 15 | $\$$ | 3 |
| 2 | 7 | $\$$ | 11 | 9 | 3 | 2 | $\$$ | 3 | 1 | 7 | 9 | 5 |
| 16 | 1 | 7 | $\$$ | 2 | 16 | 8 | 3 | $\$$ | 17 | 1 | 0 | 12 |
| 23 | 7 | 11 | 5 | $!$ | $\$$ | 23 | 7 | 3 | $\$$ | 16 | 5 | $\$$ |
| 19 | 8 | 0 | 20 | $\$$ | 22 | 0 | 16 | $\$$ | 5 | 0 | 23 | 24 |
| 7 | 6 | $\$$ | 4 | 3 | $\$$ | 12 | 22 | $\$$ | 7 | 11 | 9 | 4 |
| 23 | 4 | 8 | 10 | $\$$ | 16 | 5 | $\$$ | 3 | 2 | 7 | $\$$ | 7 |
| 25 | 9 | 21 | 3 | $\$$ | 8 | 16 | 11 | 12 | 7 | 1 | $\$$ | 0 |
| 15 | $\$$ | 17 | 0 | 5 | 4 | 3 | 4 | 24 | 7 | $\$$ | 6 | 4 |
| 24 | 4 | 5 | 0 | 1 | 5 | $\$$ | 0 | 15 | $\$$ | 3 | 20 | 0 |
| $\$$ | 3 | 2 | 0 | 16 | 5 | 9 | 8 | 6 | $\$$ | 9 | 8 | 6 |
| $\$$ | 3 | 2 | 4 | 1 | 3 | 7 | 7 | 8 | . | $\$$ | 2 | 9 |
| 17 | 17 | 22 | $\$$ | 8 | 7 | 20 | $\$$ | 22 | 7 | 9 | 1 | $!$ |



## Calendar

Session 11: (all dates in 2012)

| September | 13 | Start of the eleventh session! |
| :--- | :---: | :--- |
|  | 20 |  |
| October | 27 | Charlene Morrow, Mt. Holyoke |
|  | 4 |  |
|  | 11 | Pardis Sabeti, Broad Institute/Harvard |
|  | 18 |  |
| November | 25 | Anoush Najarian, MathWorks |
|  | 1 |  |
|  | 8 |  |
|  | 15 |  |
|  | 22 | Thanksgiving - No meet |
|  | 29 |  |
| December | 6 |  |

Session 12: (all dates in 2013)
January $\quad 31 \quad$ Start of the tenth session!

February 7
14 Emily Riehl, Harvard University
21 No meet
28
March 7
14
21 No meet
28
April 4 Crystal Fantry, Wolfram Research
11
18 No meet
25
May
2
9

Spring Math Contest Prep begins January 27, 2013. For more information, please visit www.girlsangle.org/page/contest_prep_FAQ.html.

## Girls’ Angle: A Math Club for Girls

## Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

What is Girls' Angle? Girls' Angle is a math club for girls and a supportive community for all girls and women engaged in the study, use and creation of mathematics. Our primary mission is to foster and nurture girls' interest and ability in mathematics and empower them to be able to tackle any field, no matter the level of mathematical sophistication required. We offer a comprehensive approach to math education and use a four component strategy to achieve our mission: Girls' Angle mentors, the Girls' Angle Support Network, custom content production including our magazine, the Girls’ Angle Bulletin, and various outreach activities such as our Math Treasure Hunts and Community Outreach.

Who are the Girls’ Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls’ Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is the Girls' Angle Bulletin? The Girls' Angle Bulletin is a bimonthly (6 issues per year) publication that features interviews, articles and information of mathematical interest. The electronic version is free. The print version (beginning with volume 3, number 1) costs $\$ 36$ for an annual subscription and brings with it access to our mentors through email. Subscribers may send us their solutions, questions, and content suggestions, and expect a response. The Bulletin targets girls roughly the age of current members. Each issue contains a variety of content at different levels of difficulty extending all the way to the very challenging indeed.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We also aim to overcome math anxiety and build solid foundations, so we welcome all girls regardless of perceived mathematical ability. There is no entrance test.

How do I join? Membership is granted per session. Members have access to the club where they work directly with our mentors exploring mathematics. You can also pay per meet, but it is slightly more expensive. We currently operate in 12 meet sessions, but girls are welcome to join at any time. The program is individually focused so the concept of "catching up with the group" doesn't apply. If you cannot attend the club, you can purchase a Remote Membership which comes with a year-long subscription to the Bulletin. Remote members may email us math questions (although we won't do people's homework!), send us problem solutions for constructive comment, and suggest content for the Bulletin. To become a remote member, you can simply subscribe to the print version of the Bulletin.

Where is Girls' Angle located? Girls' Angle is located about 12 minutes walk from Central Square on Magazine Street in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls’ Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls’ Angle tax deductible? Yes, Girls’ Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to Girls’ Angle and send to Girls’ Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls’Angle director? Ken Fan is the director and founder of Girls’ Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Pierce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science and worked in the mathematics educational publishing industry. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum. These experiences have motivated him to create Girls’ Angle.

Who advises the director to ensure that Girls’ Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:<br>Connie Chow, executive director of Science Club for Girls<br>Yaim Cooper, graduate student in mathematics, Princeton<br>Julia Elisenda Grigsby, assistant professor of mathematics, Boston College<br>Kay Kirkpatrick, assistant professor of mathematics, University of Illinois at Urbana-Champaign<br>Grace Lyo, Moore Instructor, MIT<br>Lauren McGough, MIT ' 12<br>Mia Minnes, SEW assistant professor of mathematics, UC San Diego<br>Beth O'Sullivan, co-founder of Science Club for Girls.<br>Elissa Ozanne, assistant professor, UCSF Medical School<br>Kathy Paur, Kiva Systems<br>Bjorn Poonen, professor of mathematics, MIT<br>Gigliola Staffilani, professor of mathematics, MIT<br>Bianca Viray, Tamarkin assistant professor, Brown University<br>Lauren Williams, assistant professor of mathematics, UC Berkeley

At Girls’ Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle any field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

## Girls’ Angle: A Math Club for Girls

## Membership Application


#### Abstract

Applicant's Name: (last) $\qquad$ (first) $\qquad$ Applying For (please circle): Membership Remote Membership/Bulletin Subscription Parents/Guardians: $\qquad$


Address: $\qquad$ Zip Code: $\qquad$
Home Phone: $\qquad$ Cell Phone: $\qquad$ Email: $\qquad$
For membership applicants only, please fill out the information in this box.
Emergency contact name and number: $\qquad$
Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. They will have to sign her out. Names: $\qquad$
Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? Yes No

Eligibility: For now, girls who are roughly in grades 5-11 are welcome. Although we will work hard to include every girl no matter her needs and to communicate with you any issues that may arise, Girls' Angle has the discretion to dismiss any girl whose actions are disruptive to club activities.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

Date: $\qquad$
(Parent/Guardian Signature)
Membership-Applicant Signature: $\qquad$
$\square$ Enclosed is a check for (indicate one) (prorate as necessary)
$\$ 216$ for a one session Membership (which includes 12 two-hour club meets)
$\square \$ 36$ for a one year Remote Membership (which includes 1-year subscription to Bulletin)I am making a tax free charitable donation.

I will pay on a per meet basis at $\$ 20 /$ meet. (Note: You still must return this form.)
Please make check payable to: Girls' Angle. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle @gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

# Girls’ Angle: A Math Club for Girls Liability Waiver 

I, the undersigned parent or guardian of the following minor(s)
do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: $\qquad$ Date: $\qquad$
Print name of applicant/parent: $\qquad$
Print name(s) of child(ren) in program: $\qquad$

A Math Club for Girls


[^0]:    ${ }^{1}$ A polygon can be cut up into a finite number of pieces and rearranged to form any other polygon of the same area. This is not true of polyhedra. If you're interested in learning more about this, look up Hilbert's $3^{\text {rd }}$ problem and Dehn invariants.

[^1]:    ${ }^{1}$ This content supported in part by a grant from MathWorks.

