## Girls' Bulletin

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To Foster and Nurture Girls' Interest in Mathematics



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## From the Founder

If you feel that math is too rigid because there's only one correct answer, try switching things around and create the problem instead! Suddenly, it may seem that the possibilities are limitless. That can also be paralyzing. Seek that place where math flows within. -Ken Fan, President and Founder


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## Girls’ Angle Bulletin

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## Girls' Angle: A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

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[^0]
## An Interview with Karen Lange, Part 3

This is the third part of our four part interview with Prof. Karen Lange of Wellesley College.

At the end of Part 2, Prof. Lange asked us to come up with our own answer to the question, "What do we mean to compute?"

Karen: Every schoolkid knows what it means to do a computation, intuitively. They've done a lot of them, but if you try and write down what is computation, that's kind of tricky. It was a little less than 100 years ago that Alan Turing and others were wrestling with that exact philosophical question: What do we mean by computation exactly? And they were wrestling with exactly, precisely what should computation mean?

And now, I think, most people have accepted that Alan Turing came up with this idea for what computation is. He was coming to answer a philosophical question. People are always like, "Oh, math and beauty, or math philosophy, what does it do for us?" It gave us computers! In that sense, it was coming from a truth and beauty place.

So, once you have an idea - once you formalize what it means to compute if you're an engineer, you can start to figure out how to actually build the machine. Obviously, that's been done, but on the philosophical side, you can start saying to yourself, "What can computers not do?" And that's exactly where computability theory starts.

There are lots of things I can program a computer to do. Consider the problem: I give you a whole number. Is it prime or not? I can write a computer
"Math and beauty, or math philosophy, what does it do for us?" It gave us computers! In that sense, it was coming from a truth and beauty place.
program for it. For example, I could systematically go through the positive integers starting at 2 and checking if the number is divisible by that positive integer. If it gets to the square root of the number without finding a factor, then the computer says, "The number is prime." Otherwise, it says, "The number is not prime." This is not a fast algorithm. The computer scientists have written much faster programs, but I could just factorize it. It might take a while, but I can do it.

But a natural question would be to ask, "Are there problems, like a yes/no problem, where if you give me a natural number, I cannot write a computer program to give you the answer?"

And the first cool fact of computability theory is that there are lots of non-computable problems. There's lots of math problems that computers cannot solve in this way. Now, you might say, "How do you know that? Give me some examples." And I'd be happy to do that, but I just can't help making the connection to what we've been talking about, because it's so awesome.

We've talked about how the rational numbers are countable, and it's not too hard to think about the idea that you can list out all the computer programs in the world. Your list is going to have some really crummy programs in it, but I don't care about that. I just want a list of all the programs in the world.

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President and Founder
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## Romping Through the Rationals, Part 7 <br> by Ken Fan I edited by Jennifer Sidney

Emily: Let's see if we can prove that any two finite rational rompers that represent the same rational numbers and begin and end with the same number can be transformed into each other via a finite sequence of swap operations. For inductive purposes, suppose they agree on the first $n-1$ terms but differ at the $n$th term, like this:

$$
\begin{aligned}
& a_{1}, a_{2}, a_{3}, \ldots, a_{n-1}, a_{n}, \ldots, a_{N} \\
& a_{1}, a_{2}, a_{3}, \ldots, a_{n-1}, b_{n}, \ldots, b_{N}
\end{aligned}
$$

where $a_{n} \neq b_{n}$, but $a_{N}=b_{N}$. Let's again call the second sequence the "target" sequence.

Jasmine: Now we know that any number $x$ that occurs in one of the sequences must also occur in the other; furthermore, $x$ appears precisely the same number of times in both sequences.

Emily and Jasmine are studying sequences $a_{n}$ of nonnegative integers that have the property that consecutive terms are relatively prime and every nonnegative rational number is equal to $a_{n} / a_{n+1}$ for a unique $n$. They have dubbed these sequences "rational rompers."

After finding a counterexample to their latest conjecture, they expanded their "splice" operation to a new operation that they call a "swap." In thinking about swaps, they ended up proving that any two finite rational rompers that begin and end the same way and represent the same rational numbers must contain the same numbers, and each number that appears must appear precisely the same number of times in both sequences.

Armed with this knowledge, will they be able to prove their conjecture?

Emily: Since the two sequences are identical for the first $n-1$ terms and also on the last term, every number $x$ also appears in the subsequence $a_{n}, \ldots, a_{N-1}$ precisely the same number of times that $x$ appears in the subsequence $b_{n}, \ldots, b_{N-1}$.

Jasmine: Also, $a_{n-1}$ and $b_{n}$ appear consecutively $-a_{n-1}, b_{n}$ - in the first sequence somewhere after the ( $n-1$ )-th term, since the two sequences represent the same rational numbers.

Emily: The situation where $a_{n}=a_{n-1}$ and $a_{n+1}=b_{n}$ feels slightly different to me from the case where $a_{n-1}, b_{n}$ appears in the first sequence somewhere after the $n^{\text {th }}$ term, so I'd like to handle those cases separately.

Jasmine: Okay, let's think about the case where $a_{n}=a_{n-1}$ and $a_{n+1}=b_{n}$, then. In that case, the first sequence goes

$$
a_{1}, a_{2}, a_{3}, \ldots, a_{n-1}=1, a_{n}=1, a_{n+1}=b_{n}, a_{n+2}, \ldots, a_{N} .
$$

To swap that $b_{n}$ so that it becomes the $n^{\text {th }}$ term, we need a 1 to occur after the $(n+1)$-th term.
Emily: Let's see if counting 1's will guarantee that such a 1 exists.

Jasmine: Okay. If no such 1 exists, then the total number of 1 's that occur from term $n$ to term $N$ in the first sequence is just one. And this would have to be true of the target sequence, too.

Emily: But that's impossible! Since the first sequence represents the rational number 1 as the ratio of $a_{n-1}$ and $a_{n}$, the target sequence must also represent 1 . But we know that none of the consecutive ratios $a_{k} / a_{k+1}$ equal 1 for $k<n-1$, and we know $a_{n-1} / b_{n}$ is not 1 since $b_{n}$ can't be 1 , so there must be two consecutive 1's somewhere among the terms from $n$ to $N$ in the target sequence. That contradicts the fact that the number of 1 's must be the same in both sequences!

Jasmine: Super! So now let's assume that $a_{m}=a_{n-1}$ and $a_{m+1}=b_{n}$, where $m>n$, so our first sequence goes like this:

$$
a_{1}, a_{2}, a_{3}, \ldots, a_{n-1}, a_{n}, \ldots, a_{m}=a_{n-1}, a_{m+1}=b_{n}, \ldots, a_{N} .
$$

What we hope for is that one of the $a_{k}$ for $n \leq k \leq m$ reappears after the ( $m+1$ )-th term, for then we could perform a swap and move the $b_{n}$ located in position $m+1$ to position $n$, completing our inductive argument.

Emily: Let's assume that doesn't happen. That is, for each $k$ between $n$ and $m$, inclusive, let's assume that $a_{k}$ does not appear again after the $(m+1)$-th term. If what we hope for is true, this assumption should lead to a contradiction.

Jasmine: Hey Emily, the first $n-1$ terms of the two sequences are identical and don't seem to play a role in our analysis. So, to simplify things, let's just assume that $n=2$, and that both sequences start with the same number, then differ on their second terms.

Emily: I think you're right, so let's make that simplification. Then our two sequences go:

$$
\begin{gathered}
a_{1}, a_{2}, \ldots, a_{m}=a_{1}, a_{m+1}=b_{2}, \ldots, a_{N} \\
a_{1}, b_{2}, \ldots, b_{N}
\end{gathered}
$$

where $m>2$ and $a_{N}=b_{N}$, but $a_{2} \neq b_{2}$.

Jasmine: And we assume that none of the numbers $a_{2}, \ldots, a_{m}$ appear after the ( $m+1$ )-th term in the first sequence.

Emily: Actually, would you mind if, for the moment, we also assume that $m=3$ ? I realize that this will not always be the case; but if what we're trying to prove is true, we'd certainly have to be able to prove it if $m=3$, which might be easier to prove anyway.

Jasmine: That's fine with me. In the case $m=3$, we are hoping that either $a_{1}$ or $a_{2}$ appears after the $4^{\text {th }}$ term in the first sequence, so we assume that does not happen. What can we say?

Emily: Since $a_{2}$ is not equal to either $a_{1}$ or $b_{2}$, that would mean that $a_{2}$ appears exactly once in both sequences.

Jasmine: So somewhere after the $2^{\text {nd }}$ term of the target sequence, there occurs $a_{2}$. And since $a_{1} / a_{2}$ and $a_{2} / a_{1}$ are represented numbers, that occurrence of $a_{2}$ must be sandwiched between two occurrences of $a_{1}$.

Emily: But if $a_{1} \neq b_{2}$, that would mean the target sequence contains three $a_{1}$ 's, whereas the first sequence would only contain two $\ldots$ a contradiction!

Jasmine: The only way $a_{1}$ could equal $b_{2}$ is if $a_{1}=b_{2}=1$ and the first sequence begins $1, a_{2}, 1,1, \ldots$. But then we'd be able to perform a swap that moves the $a_{2}$ located in position 2 to in between the consecutive 1 's, resulting in agreement with the target sequence on the first two terms!

Emily: Nice! That settles the situation when $m=3$.
Jasmine: Onto the case $m=4$ ! So now the first sequence goes

$$
a_{1}, a_{2}, a_{3}, a_{4}=a_{1}, a_{5}=b_{2}, \ldots, a_{N}
$$

and we're hoping to find another occurrence of $a_{2}, a_{3}$, or $a_{4}=a_{1}$ after the $5^{\text {th }}$ term (in this first sequence). So we assume this does not happen.

Emily: We've handled the case where $a_{1}=a_{2}$, so we can assume those aren't equal; but it seems like $a_{3}$ could be equal to either $a_{1}$ or $a_{2}$, which would affect the tallies of how many times $a_{2}, a_{3}$, and $a_{4}$ occur.

Jasmine: All of these cases make it a challenge to keep things organized! But let's forge ahead and look at the three possibilities separately: either $a_{3}=a_{1}, a_{3}=a_{2}$, or $a_{3}$ is neither $a_{1}$ nor $a_{2}$. If $a_{3}=a_{1}$, the sequence actually goes $1, a_{2}, 1,1, a_{5}=b_{2}, \ldots, a_{N}$. So 1 appears three times, and $a_{2}$ appears once. The target sequence begins with $1, b_{2}$, so there must be two more 1 's in that sequence, and they must appear consecutively since both sequences represent the rational number 1. The first sequence also represents $1 / a_{2}, a_{2} / 1$, and $1 / b_{2}$, so the term before the two consecutive 1 's in the target sequence must be $a_{2}$, and the term after must also be $a_{2}$. Hey! That's two appearances of $a$ - a contradiction!

Emily: Yay! And if $a_{3}=a_{2}$, the first sequence goes $a_{1}, 1,1, a_{1}, b_{2}, \ldots, a_{N}$. So both $a_{1}$ and 1 appear twice. The target sequence must go $a_{1}, b_{2}, \ldots$, and somewhere after the $2^{\text {nd }}$ term, there must occur consecutive 1's. These 1's must be sandwiched between two occurrences of $a_{1} \ldots$ but that means that $a_{1}$ must appear at least three times in the target sequence - a contradiction!

Jasmine: This counting strategy seems to be working! Now, what about when $a_{3}$ is neither $a_{1}$ nor $a_{2}$ ? In that case, $a_{1}$ appears twice, and $a_{2}$ and $a_{3}$ each appear just once.

Emily: Wait. Isn't it possible that $a_{3}=b_{2}$, in which case $a_{3}$ appears twice?
Jasmine: Oh, good catch. More cases! Well, either way, there must be an occurrence of $a_{1}$ after the $2^{\text {nd }}$ term in the target sequence, and that occurrence of $a_{1}$ must be preceded by $a_{3}$ and followed by $a_{2}$. So if $a_{3} \neq b_{2}$, the target sequence must go $a_{1}, b_{2}, \ldots, a_{3}, a_{1}, a_{2}, \ldots$. And since
$a_{2} / a_{3}$ is represented and $a_{2}$ only appears once, there must be an $a_{3}$ after that $a_{2}$. So after the second term, there would be two occurrences of $a$ - a contradiction! And if $a_{3}=b_{2}$, then the first sequence begins $a_{1}, a_{2}, a_{3}=b_{2}, a_{1}, a_{3}=b_{2}, \ldots$, and the target sequence would have to go $a_{1}, b_{2}=a_{3}, a_{1}, a_{2}, a_{3}, \ldots . \mathrm{Hm}$. I don't see a contradiction.

Emily: I guess that is a real possibility. But in that case, we can perform a swap. We move the $a_{2}$ located in position 2 to be in between the $4^{\text {th }}$ and $5^{\text {th }}$ terms, transforming the first sequence to the sequence $a_{1}, a_{3}=b_{2}, a_{1}, a_{2}, a_{3}, \ldots$, which agrees with the target sequence on the first two terms; in fact, on the first five terms! So that case works out, too!

Jasmine: Cool!

Emily: Gosh, it's looking like our swapping conjecture is true. But how are we going to prove it for all $m>2$ ? Each time we increase $m$, the number of cases we need to consider grows.

Jasmine: And the $m=4$ case shows us that in some cases, we'll find a contradiction via counting, but in other cases, we may not actually get a contradiction and, instead, have to find a swap operation that does what we want.

Emily: Should we look at the $m=5$ case?

Jasmine: Well, let's say we do that: we carefully cover all of the cases, which seem to be determined by various equalities between the terms, and we find out that, yes, that case works, too. That exercise will only help us if it suggests a more general observation that we would be able to apply to any value of $m$.

Emily: Yes, because if we don't make such an observation, I guess we'd be in the same situation we're in now, except that we'd be wondering whether to look at the $m=6$ case.

Jasmine: Instead of the case $m=5$, let's take a bird's-eye view of the general situation and see if we can notice something about it that is more general.

## Emily: Okay.

Jasmine: So, we've got two sequences:

$$
\begin{gathered}
a_{1}, a_{2}, \ldots, a_{m}=a_{1}, a_{m+1}=b_{2}, \ldots, a_{N} \\
a_{1}, b_{2}, \ldots, b_{N}
\end{gathered}
$$

where $m>4$ and $a_{N}=b_{N}$. Also, $a_{2} \neq b_{2}$. And we assume that none of the numbers that appear among the first $m$ terms reappears after the $(m+1)$-th term. We hope to get either a contradiction or find a swap operation that can transform the first sequence to one that agrees with the target sequence on the first two terms. What can we say?

Emily and Jasmine think.

Emily: Somehow, we want to use the fact that any number $x$ appears the same number of times in both sequences. The myriad cases begin to pop up as we try to keep track of which terms equal which terms, because such equalities change the tallies for the number of times each number appears in the sequence.

Jasmine: Yes, and when $m$ is large, that could lead to a lot of different cases!
Emily: Maybe we can finesse these cases away by not worrying about any equalities between the first $m$ terms. After all, we don't really have to count the number of times any particular number appears; we just have to use the fact that every number appears the same number of times in both sequences.

Jasmine: How do you mean?
Emily: It's like when you have two sets and want to know if they have the same number of elements. You don't need to actually count the number of elements in each one and see if the counts come out equal. Instead, you can just pair them up and see if each element in one set gets paired uniquely with an element in the other, and vice versa.

Jasmine: I see where you're going.

Emily: In our situation, we know that we can pair each of the terms $a_{1}$ through $a_{m}$ with a different term in the target sequence in such a way that each term is paired with a term of the same value. That is, we can pair up the first term in the first sequence, $a_{1}$, with the first term $a_{1}$ in the target sequence, and we can pair the second term in the first sequence, $a_{2}$, with some occurrence of $a_{2}$ in the target sequence, and so on. Thus, we can define a function $j$ which maps an index $k$ between 1 and $m$, inclusive, to an index $j(k)$ such that $b_{j(k)}=a_{k}$, and in such a way that $j(k)=j(l)$ if and only if $k=l$.

Jasmine: I like this idea! And since that accounts for all occurrences of the numbers that appear among the first $m$ terms of the first sequence ...

Emily: ... unless $b_{2}$ happens to be equal to one of the first $m$ terms ...

Jasmine: Oh yeah, you're right. I keep forgetting about that possibility! Actually, why don't we just go ahead and extend the domain of $j$ to all $k$ such that $1 \leq k \leq m+1$, and then go ahead and insist that $j(1)=1$ and $j(m+1)=2$. This way, we can, in fact, say that $j(k)>2$ for all $2 \leq k \leq m$, and any index $z$ not in the range of $j$ must hold a value $b_{z}$ that is different from any of the first $m$ terms of the first sequence.

Emily: Okay, that seems like a good piece of housekeeping.
Jasmine: Wait a second!
Emily: What?

Jasmine: Doesn't it have to be that if $2 \leq k \leq m$, then $b_{j(k)}$ must be preceded and followed only by other terms that have been paired? Because if $b_{j(k)}$ is preceded or followed by, say, $x$, where $x$ is not a number that appears among the first $m$ terms of the first sequence, it means that the rational number $x / b_{j(k)}$ or $b_{j(k)} / x$ must be represented by the sequence. But this number is not represented by the first sequence, because the only terms equal to $b_{j(k)}$ in the first sequence must occur among the first $m$ terms.

Emily: That sounds correct. The only value of $k$ I' $m$ unsure about is when $k=m$. Let me think that through. If $k=m$, then $b_{j(k)}=a_{1}$. In the first sequence, there could occur an $a_{1}$ followed by any term among the first $m$ terms, or it can be followed by $b_{2}$. If it's not followed by $b_{2}$, then what you said is correct. So the question is, could it be that $b_{j(m)+1}=b_{2}$ ? Ah, I see; that's not possible because the first two terms of the target sequence are $a_{1}$ and $b_{2}$, so $a_{1} / b_{2}$ is already represented there, therefore cannot be represented again. Thus $b_{j(m)+1}$ must also be a number that appears among the first $m$ terms of the first sequence. So, yes, I agree with you!

Jasmine: I think I'm seeing a proof ...
Emily: Me too! Because what you just said means that the indices $j(k)$ for $2 \leq k \leq m$ must be clumped together; otherwise, one of them would be next to a number that isn't paired up, and that would contradict the fact that the sequences represent the same rational numbers!

Jasmine: Yes, and that clump of indices cannot be immediately preceded or followed by a number that isn't paired up, for the exact same reason.

Emily: That means that if none of the numbers that appear among the first $m$ terms of the first sequence appear again after the $(m+1)$-th term, the first sequence can only be $m+1$ terms long and $b_{2}$ must occur amongst the first $m$ terms of the first sequence. That is, the first sequence must look like this:

$$
a_{1}, a_{2}, \ldots, a_{k}=b_{2}, \ldots, a_{m}=a_{1}, a_{m+1}=b_{2},
$$

for some $k$ that satisfies $2<k<m+1$.
Jasmine: But in that case, we can move the block of terms from the $2^{\text {nd }}$ through the $(k-1)$-th term and situate it between the $m$ and $(m+1)$-th term via a single swap maneuver!

Emily: That's it! It works!

Jasmine: High five! We've proven that any two rational rompers can be transformed from one to the other by a sequence (possibly infinite) of swaps; furthermore, two rational rompers differ in finitely many places if and only if one can be transformed to the other via a finite sequence of swaps. Indeed, any two finite rational rompers that represent the same rational numbers and have the same first terms and the same last terms can be transformed into each other via a finite sequence of swaps.

Emily: Hey Jasmine, I think it's time to swap some cash for another banana bonanza!

## Optimal Resource Placement: From Disneyland to Dominating Sets, Part 1

by Jillian Cervantes and Pamela E. Harris ${ }^{12}$

Legend has it that when Walt Disney designed the Disneyland park in the 1950s, he was remarkably intentional about the placement of trash cans [1]. Disney postulated that as guests walked around the park, they would carry their garbage for a maximum of 30 feet before dropping it on the ground. To maintain the cleanliness of his theme park, Disney decreed that trash cans should be placed throughout the space so that a Disneyland guest is never more than 30 feet away from the nearest receptacle.

In 2010, about sixty years after the opening of Disneyland, the first mass-market electric cars became available. There arose a phenomenon amongst electric vehicle drivers known as "range anxiety," which is the fear that one's electric vehicle will run out of battery power before reaching the next charging station [5]. As a result, governments who wished to incentivize electric vehicle usage had to think about the optimal placement of charging stations within a given area. The average range of an electric car battery is 350 kilometers, so government bodies realized that they needed to ensure drivers were never more than 350 kilometers away from the nearest charger.

What do these two problems have in common? Whether you're choosing the placement of Disneyland trash cans or electric vehicle charging stations, it's imperative that you meet your specific distance requirements ( 30 feet for the trash cans, 350 kilometers for the charging stations). But installing these amenities costs money. How can you be certain that you're installing the optimal number of resources? In other words, how can you minimize the number of resources used while still meeting the strict distance requirements?

The answer lies in the mathematical concept of a dominating set. In order to understand dominating sets, let's first learn a bit about graphs.

In the context of graph theory, a graph is a structure defined by two sets. The members of one set are called vertices and the members of the other set are called edges. For our purposes, an edge is a set consisting of two distinct vertices, and we say that the edge connects those two vertices. ${ }^{3}$

Let's take a look at the grid graph with 3 rows and 4 columns of vertices.


[^1]We denote this type of grid graph according to the number of vertices in each column and row. The graph above would be denoted $3 \times 4$. (It has 12 vertices and 17 edges.)

We now define a dominating set of a graph. A dominating set is a set of vertices that satisfy the property that every vertex in the graph is either in the set, or a distance of 1 away from a vertex in the set. Here, the distance between two vertices of a graph is the minimum number of edges that need to be traversed to travel from one vertex to the other along edges of the graph. So a distance of 1 between two vertices means that the vertices are separated by one edge. For example, these two sets of circled vertices each dominate the $3 \times 4$ graph:


But the set below does not, as vertices $v_{1}$ and $v_{2}$ are neither selected nor neighboring a selected vertex:


Given a graph $G$, a natural question to ask is, what is the minimum number of vertices that can dominate the graph? This number is called the domination number, and is denoted $\gamma(G)$.

Exercise. Consider the graph $3 \times 4$ and determine if the domination number is 3 . Why or why not?

The formal study of graph domination began in 1962, when Claude Berge published a book on graph theory [2], which included the domination problem. However, math questions related to domination were being asked nearly 100 years prior, through the mathematical study of chess! ${ }^{4}$

In 1862, C.F. De Jaenisch tried to find the minimum number of queens required to cover, or dominate, an $n \times n$ chess board [8]. In this instance, to dominate a chess board is to place pieces such that all vacant squares can be attacked at least once. Shown at right is a solution for the classic $8 \times 8$ chess board. (How would you realize this chess question in terms of finding a dominating set for a graph? What would the vertices and edges of the graph be?)


[^2]In 1975, Cockayne and Hedetniemi published a survey paper on domination of graphs, and this spurred an increased interest in the subject [4]. Since then, over 2000 papers have been published on domination theory. ${ }^{5}$
You may be asking, how can there be over 2000 original publications on this one specific subject? Well, a wonderful thing about graph theory is that it is highly applicable and it has many parameters that can be adjusted to create new problems. Let's take a tour of some important mathematical results over the history of domination theory. In 1983, Michael Jacobson and Lael Kinch gave the domination numbers for $2 \times n, 3 \times n$, and $4 \times n$ grid graphs [7]. In his 1992 PhD thesis [3], Tony Yu Chang proved the following inequality for $n \geq m \geq 8$, and conjectured that it was actually an equality for $n \geq m \geq 16$ :

$$
\gamma(m \times n) \leq\left\lfloor\frac{(m+2)(n+2)}{5}\right\rfloor-4,
$$

where $\lfloor x\rfloor$ denotes the floor of $x$, i.e., greatest integer less than or equal to $x$.
Chang's conjecture remained unsolved for nearly twenty years! His formula was proved to be an equality for $n \geq m \geq 16$ in 2011 by Gonçalves, Pinlou, Rao, and Thomassé, who gave a computer-aided proof [6].

Why was this conjecture so difficult to prove? It turns out that proving a lower bound for the domination number is much harder than proving an upper bound. This is because proving an upper bound only requires showing that the proposed bound does indeed dominate the graph. On the other hand, proving a lower bound requires demonstration that any number of vertices smaller than the proposed bound cannot possibly dominate the graph. Do you have any ideas for how this could be proved? We'll walk through such a proof in Part 2!

For now, let's consider an application. Imagine you are a city planner and your city is represented by a $2 \times 8$ grid graph. You have been tasked with determining the number of cell phone towers that should be placed in the city, and the eligible spots for cell towers are the vertices of your grid graph.


If you've ever been unfortunate enough to find yourself in a cellular dead zone, you understand the importance of placing enough towers to sufficiently cover your city so that no one is left without reception. As you might imagine, budgets are tight, and the city council is pressuring you to minimize the number of towers-the more towers you place, the more money the city will have to spend to install them.

[^3]You know that a cell phone user will get sufficient reception as long as they are at most distance 1 away from a tower. That means that if you place towers on the vertices of the city in such a way that each vertex is at most distance 1 from a tower, the city is guaranteed to be free of dead zones.

Activity. One option is to place towers at every vertex, but you could dominate the graph with less. Take some time to try to dominate the grid graph above with as few vertices as you can. Really do it, stop reading here!

Challenge Activity. What is the smallest number of towers you have been able to place on the map which dominate the map? How do we know this is the smallest? If you are convinced that your number of towers is as small as possible, in how many distinct ways could you place that number of towers to still dominate the map?

We'll give solutions in the next installment where we'll discuss the following theorem proven by Jacobson and Kinch in 1983 [7].

Theorem. We have $\gamma(2 \times n)=\left\lceil\frac{n+1}{2}\right\rceil$.
Here, $\lceil x\rceil$ denotes the ceiling function, i.e., the smallest integer greater than or equal to $x$.

## References

[1] Beaven, Johanna. "The Art of Disney's Trash Cans." Disney World, 6 February 2020, www.wdwinfo.com/disney-merchandise/the-art-of-disneys-trash-cans/. Accessed 28 November 2023.
[2] Berge, Claude. The Theory of Graphs. Methuen, 1962.
[3] Chang, Tony Yu. Domination numbers of grid graphs. 1992. Thesis (Ph.D.)-University of South Florida.
[4] Cockayne, E. J., and Stephen J. Hedetniemi. "Optimal domination in graphs." IEEE Transactions on Circuits and Systems, vol. 22, no. 11, 1975, pp. 855-857. IEEE Xplore.
[5] "Electric car range and 5 reasons why your range anxiety is unwarranted [2023]." EVBox I Blog, 19 June 2023, blog.evbox.com/eliminate-range-anxiety. Accessed 28 November 2023.
[6] Gonçalves, Daniel, et al. "The domination number of grids." SIAM Journal of Discrete Mathematics, vol. 25, no. 3, 2011, pp. 1443-1453. arxiv.
[7] Jacobson, Michael, and Lael Kinch. "On the domination number of products of a graph; I." Ars Combinatoria, vol. 18, 1984, pp. 33-44. Science Direct.
[8] Tarr, Jennifer M. "Domination in Graphs." USF Tampa Graduate Theses and Dissertations, 2010. digitalcommons.usf.edu/etd/1786.

## Path Counting for Partitions

by Robert Donley ${ }^{6}$

edited by Amanda Galtman
We continue our investigation of partitions, Young diagrams, and the Young lattice. In the previous installment, we defined a partial order on the set of partitions and considered methods for counting chains using an up operator. Here we give a direct method, known as the hook length formula, for computing these numbers. Young diagrams play a crucial role, both in calculations and in the recording of data.

We keep the definitions and notation from previous installments; as usual, partitions are denoted by non-increasing strings of digits. In particular, some exercises in this part require the Hasse diagrams used in the previous part. Conveniently, $L(4,3)$ appears on the previous issue's cover.

Definition: Suppose $m, n \geq 1$. Denote by $L(m, n)$ the set of all partitions $\lambda$ with at most $m$ parts such that each part $\lambda_{i} \leq n$. Alternatively, the corresponding Young diagrams fit inside a rectangle with width $n$ and height $m$.

Definition: Recall that we say $\lambda$ covers $\mu$ if $\mu \leq \lambda$ and there are no other elements between $\mu$ and $\lambda$. For partitions, $\lambda$ has the same parts as $\mu$ except in one entry. For the corresponding Young diagrams, $\lambda$ is obtained by adding one square to $\mu$.

For lattice path counting, we count (saturated) chains from 0 to $\lambda$. These are sequences from 0 to $\lambda$ such that each element is covered by the next element in the sequence.

Definition: Let $f^{\lambda}$ denote the number of chains from 0 to the partition $\lambda$.
In the previous installment, we represented chains in $L(m, n)$ using certain Gelfand-Tsetlin patterns. Young diagrams give another way to represent chains, although the data is organized quite differently. Consider the following chain from 0 to 321 in $L(3,3)$ :

$$
0 \rightarrow 1 \rightarrow 2 \rightarrow 21 \rightarrow 31 \rightarrow 32 \rightarrow 321
$$

If we represent each partition as a Young diagram, then we can label the squares in the order we add them, and the last diagram encodes all information needed to recreate each step of the chain.

$$
0 \rightarrow \begin{aligned}
& 1 \\
& 0
\end{aligned} \rightarrow \begin{array}{|l|l|}
\hline 1 & 2
\end{array} \rightarrow \begin{array}{|l|l}
1 & 2 \\
\hline 3 & \rightarrow \begin{array}{|l|l|l}
\hline 1 & 2 & 4 \\
\hline 3 &
\end{array} \rightarrow \begin{array}{|l|l|l|}
\hline 1 & 2 & 4 \\
\hline 3 & 5 &
\end{array} \rightarrow \begin{array}{|l|l|l|}
\hline 1 & 2 & 4 \\
\hline 3 & 5 & \\
\hline 6 &
\end{array} \\
\end{array}
$$

Thus, chains are determined by special fillings of Young diagrams.
Definition: A standard Young tableau is a Young diagram with $k$ squares, filled with the numbers $1, \ldots, k$ such that these numbers increase along each row and column.

Exercise: Choose four other chains from 0 to 321 in $L(3,3)$ and record the corresponding standard Young tableaux.

[^4]Exercise: Reconstruct the paths in $L(2,3)$ with the following diagrams:

| 1 | 2 | 3 |
| :--- | :--- | :--- |
| 4 | 5 | 6 |


| 1 | 3 | 5 |
| :--- | :--- | :--- |
| 2 | 4 | 6 |


| 1 | 2 | 5 |
| :--- | :--- | :--- |
| 3 | 4 | 6 |

To interpret these tableaux, we recall the notion of a ballot sequence from the installment "Central Binomial Coefficients and Catalan Numbers" (see Volume 15, Number 5 of the Girls’ Angle Bulletin). Consider an election with $k$ voters and $n$ candidates. The results of the election are recorded by a Young diagram with $k$ squares and $n$ rows; the length of each row records the number of ballots cast for each candidate, and the candidates are listed in order of non-increasing ballot counts. If the largest row is unique, then the first candidate is the winner of the election. Otherwise, the election ends in a tie.

If votes are tracked and recorded one at a time in the Young diagram, then a standard Young tableau arises if the first candidate always has at least as many ballots as the second candidate, the second candidate always has at least as many ballots as the third candidate, and so on.

Exercise: Suppose six voters cast ballots in an election with candidates A, B, and C with Young diagram corresponding to 321 . Determine the ballot sequence for each of your four chains in the first exercise. For instance, the ballot sequence for the given chain to 321 is $A A B A B C$.

Another notion clarified by the use of Young diagrams are corners. We can add squares only at inside corners, which may include the spaces at the top right and lower left of the diagram. We can remove squares only from the outside corners. When we apply the up operator to a partition, it is the inside corners that are indexed. On the other hand, covering in the Hasse diagram is determined by the outside corners.

Exercise: For each element in the Hasse diagrams for $L(3,3)$ and $L(3,4)$, count the inside and outside corners. Verify that these numbers match the number of inbound and outbound edges at each element.

Exercise: What are the minimum and maximum numbers of corners of each type in a Young diagram with $r$ rows and $c$ columns? Describe the Young diagrams for which these numbers attained. What does it mean if a Young diagram has only one outside corner? One inside corner? Consider the differences in $L(m, n)$ and the general Young lattice.

One key difference between Pascal's triangle and the Young lattice is covering of elements. As we recall below, covering in Pascal's triangle is entirely uniform, determined entirely by whether an element is on the boundary or not. On the other hand, corners, and therefore shape, determine covering in the Young lattice.

These ideas are useful for checking our work when constructing Hasse diagrams for $L(m, n)$. The up operator gives an algorithm to construct all partitions for a given level, and the corners can be used to guarantee that we have included all edges of the diagram.

If we want to check our work when listing all chains ending at a given level, we can set up a decision tree. From top to bottom, a decision tree simply records all possible choices at each step. To differentiate from how the Hasse diagram works, each branch of the decision tree
records the distinct chains. To determine the chain count to $\lambda$, we count branches that end in $\lambda$. For instance, the tree on the right shows all paths to the fourth level of the Young lattice.

Exercise: Verify the counts and chains in the decision tree on the right.


Chain counting gives a measure of covering in a Hasse diagram. Another way is to simply look at all elements below a given element in the diagram.

Definition: For a given $\lambda$ in $L(m, n)$, the downset $I(\lambda)$ is the poset consisting of all $\mu \leq \lambda$ in $L(m, n)$ with the induced partial order. We obtain the Hasse diagram of $I(\lambda)$ from $L(m, n)$ by erasing all nodes that are not in $I(\lambda)$ and their adjacent links.

Exercise: Draw the downsets for $\lambda=33,321$, and 111 in $L(3,4)$. For which $\lambda$ is $I(\lambda)$ a chain? For which $\lambda$ is $I(\lambda)$ vertically symmetric? Under what circumstances is the union or intersection of downsets another downset?

Exercise: Prove that any chain from 0 to $\lambda$ is contained in $I(\lambda)$.
Exercise: For the chain from 0 to 321 in $L(3,3)$ on page 18, circle $I(\lambda)$ in the Hasse diagram of $L(3,3)$ for each $\lambda$ in the chain. Compare to the decision tree with 321 in the last row.

We return to chain counting without using the up operator. First, we recall several of the chain counting formulas already available from previous installments.

Example: The simplest example occurs when $\lambda$ corresponds to a row or column of squares. That is, if $\lambda=11 \ldots 11$ or $n$, then $f^{\lambda}=1$.

Exercise: Prove that if $f^{\lambda}=1$, then $\lambda$ is given by the previous example. Under what conditions does $f^{\lambda}=2$ ? Can $f^{\lambda}$ equal any positive integer?

Definition: We call the Young diagram for $\lambda$ a hook if all squares in the diagram are the corner square, squares to its right, or squares below it. That is, a hook corresponds to a partition $(w+1) 1 \ldots 1$ with $h 1$ s for some $h, w \geq 0$.

In this case, each chain to $\lambda$ from 0 corresponds to a word that, when read left to right, indicates the order in which squares are added to the right or below. Thus, for such a hook, $f^{\lambda}=\binom{h+w}{w}$.


Exercise: Find all words in the letters $H$ and $W$ corresponding to chains from 0 to $\lambda=31$. Also, list the corresponding tableaux.

More generally, Pascal's triangle models the set of all hooks. In the Young lattice, if we keep only the nodes corresponding to hooks, a triangle results. After we choose parameters consistently, we obtain a visual proof of the formula for $f^{\lambda}$ for hooks.

Exercise: Prove that every
 hook with more than one square covers either one or two other hooks. Prove that the downset of a hook consists only of hooks. What hooks are contained in $L(m, n)$ ? Describe the shape of the downset of a hook in the Hasse diagram of hooks above.

Exercise: Prove the formula for $f^{\lambda}$ by giving a one-to-one correspondence of Pascal's triangle with the poset of hooks. This correspondence should preserve all covering relations. (Recall that nodes of Pascal's triangle are ordered pairs $(x, y)$ with nonnegative integer entries. See the installment " $n$-Cubes and Up Operators" in Volume 16, Number 2 of the Girls' Angle Bulletin for an example of a rectangular section of Pascal's triangle.)

Next, we consider a Young diagram with two lengths, $a$ and $a+b$. Each diagram of this type occurs in $L(2, a+b)$. In fact, the calculation for these $f^{\lambda}$ extends the first Catalan number formula from the installment in Volume 15, Number 5 of the Girls’ Angle Bulletin.


To calculate $f^{\lambda}$, turn Figure 3 from that installment upside down and identify the nodes on the right half of the diagram with partitions $(a+b, a)$ as seen in the previous installment. We obtain the formula for $f^{\lambda}$ as a difference of two binomial coefficients as in the first Catalan number formula.

Exercise: For $\lambda=(a+b) a$, verify that

$$
f^{\lambda}=\binom{2 a+b}{a}-\binom{2 a+b}{a-1}=\frac{b+1}{a+b+1}\binom{2 a+b}{a} .
$$

Compare with values in $L(2,3)$ from the previous installment.
Note that we obtain the Catalan number formula when $b=0$ and the formula for hooks when $a=1$. If we interpret $f^{\lambda}$ as the number of paths from $(0,0)$ to $(a+b, a)$ that never go above the line $y=x$ in the first quadrant, then the second equality calculates $f^{\lambda}$ as a fraction of all paths to $(2 a+b, a)$ in the first quadrant.

Exercise: In the difference formula, prove that the second term counts all paths from $(0,0)$ to $(a+b, a)$ in the first quadrant that go above the line $y=x$ at least once. (If you get stuck, look up the Reflection Method for the ballot problem.)

To proceed further, we give the hook length formula, although we omit the proof. The hook length formula demonstrates an advantage of Young diagrams over the partition notation, as the geometry of the diagram provides for both calculation and recording of data for the formula. Furthermore, the hook length formula gives a calculation of $f^{\lambda}$ that requires no direct use of the up operator or Hasse diagrams.

The main calculation in the formula depends on the hooks in the Young diagram.
Definition: The hook length corresponding to a square in a Young diagram is the length of the hook with that square as the corner. For the hook length formula, we record each hook length in the corresponding square.


For instance, the diagram at the right shows the case of a single hook of length 4, and the four examples below give diagrams with all hook lengths recorded. Do you notice any patterns?


| 5 | 4 | 3 | 2 |
| :--- | :--- | :--- | :--- |
| 4 | 3 | 2 | 1 |

Hook length formula for $f^{\lambda}$ : Suppose $\lambda$ is a partition of $k$, and let $P$ be the product of all hook lengths in the Young diagram corresponding to $\lambda$. Then the number of chains from 0 to $\lambda$ is

$$
f^{\lambda}=k!/ P .
$$

Exercise: Use the hook length formula to calculate $f^{\lambda}$ for the four diagrams above. Compare with the values obtained from the up operator for $L(3,4)$.

Exercise: Prove that the hook length formula agrees with the above formulas when $\lambda$ corresponds to a hook or has exactly two parts. Verify the hook length formula for all values of $L(2,3)$ directly.

At first, the hook length formula may seem
unwieldy for large diagrams, but hook lengths behave uniformly within the rectangles


| $\rightarrow$ | 16 | 15 | 14 | 11 | 10 | 9 | 8 | 5 | 4 | 3 |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 15 | 14 | 13 | 10 | 9 | 8 | 7 | 4 | 3 | 2 |  |
|  | 14 | 13 | 12 | 9 | 8 | 7 | 6 | 3 | 2 | 1 |  |
|  | 10 | 9 | 8 | 5 | 4 | 3 | 2 |  |  |  |  |
|  | 9 | 8 | 7 | 4 | 3 | 2 | 1 |  |  |  |  |
|  | 4 | 3 | 2 |  |  |  |  |  |  |  |  |
|  | 3 | 2 | 1 |  |  |  |  |  |  |  |  | sectioned by the inside corners of the diagram. To simplify computations, we can note the hook length of the square at the lower right of the rectangle and then extend by adding 1 as we move to the left or above within the rectangle. See the diagrams above.

We recall the formula for a product of positive consecutive integers.
Exercise: Fix $n \geq 1$. Prove that $n(n+1) \ldots(n+m-1)=(n+m-1)!/(n-1)!$.
Exercise: Calculate the hook length formula in the case of a rectangle with height $m$ and width $n$. Compare with the up operator calculations for $\lambda=333,444$, and 4444.

Exercise: Calculate the hook length formula in the case when the rows have only two sizes. Compare with the up operator calculations for $\lambda=4411,4422$, and 4433 in $L(4,4)$.

Let's calculate $f^{\lambda}$ for $\lambda=(a+b+c)(a+b) a$, the general case with three parts. With $a, b$, and $c$ as in the diagram at right, the above reasoning shows that the product of hook lengths is

$$
a!b!c!\frac{(a+b+1)!}{(b+1)!} \frac{(b+c+1)!}{(c+1)!} \frac{(a+b+c+2)!}{(b+c+2)!}
$$


from which we obtain the formula

$$
f^{\lambda}=\frac{(3 a+2 b+c)!(b+1)(c+1)(b+c+2)}{a!(a+b+1)!(a+b+c+2)!} .
$$

Exercise: Compare with the values calculated by the up operator for $\lambda=321,421,431$, and 432 .
Exercise: Compare with the formula for a hook ( $a=1, b=0$ ), for two parts $(a=0)$, and for rectangles with three parts ( $b=c=0$ ).

Exercise: Rewrite $f^{\lambda}$ as a product of a fraction with the multinomial coefficient $C(a+b+c, a+b, a)$ from "Compositions and Divisors" (see Volume 16, Number 3 of the Girls’ Angle Bulletin). Since the multinomial coefficient counts all chains to $(a+b+c, a+b, a)$ in the first octant of three-space, this formula expresses $f^{\lambda}$ as a fraction of all such chains.

Finally, we consider the special case of triangular Young diagrams.
Exercise: Calculate the hook length formula for $\lambda=n(n-1) \ldots 321$. What happens to the rectangular sections? What pattern appears in the product of hook lengths?

Exercise: To improve the formula from the previous exercise, prove that

$$
1 \times 3 \times 5 \times \cdots \times(2 n-1)=\frac{(2 n)!}{2^{n} n!}
$$

Exercise: Use this formula to calculate $f^{\lambda}$ for $\lambda=1,21,321,4321$, and 54321. Then, look up the associated sequence in the Online Encyclopedia of Integer Sequences (oeis.org).

## Cubics, Part 1

by Lightning Factorial I edited by Jennifer Sidney

In the spirit of figuring things out, we asked Lightning Factorial to try to find a formula for the roots of a cubic equation in terms of its coefficients. The cubic formula, like the quadratic formula, is well known and can readily be looked up. But trying to figure something out yourself can be far more fun and can lead to real adventure. Let's see what Lightning comes up with!

How can we find solutions to the cubic equation

$$
a x^{3}+b x^{2}+c x+d=0
$$

where $a, b, c$, and $d$ are fixed real numbers and $x$ is unknown?
For the record, I know that there are formulas, analogous to the quadratic formula, for the solutions to this cubic equation. But I can't remember what they are other than that they involve taking cube roots. I've been asked to see how far I can get on my own ... just for fun.

But I don't want to spoil your fun, so if you want to think about it yourself, read no further and see what you can make of it! If you figure out a way to the goal, we'd love to hear about it, so please send your way of deriving the cubic formula to girlsangle@gmail.com.

Here goes ...
The first thing I notice is that if $a=0$, then the equation is a quadratic equation (or if $b$ is also 0 , a linear equation, or if $a, b$ and $c$ are all 0 , then there's no unknown at all to solve for!). Since I know how to solve quadratic equations, I'll assume that $a$ is not 0 .

If $a$ is not 0 , I can divide the entire equation by $a$ to obtain

$$
x^{3}+(b / a) x^{2}+(c / a) x+(d / a)=0
$$

which is an equation of the form $x^{3}+B x^{2}+C x+D=0$. If I'm able to solve cubic equations where the lead coefficient is 1 , then I can solve all cubic equations. So, resetting notation, I'll focus on solving the cubic equation

$$
x^{3}+b x^{2}+c x+d=0
$$

I already feel stuck, so I guess I'll graph some cubic functions of this form.
The graphs suggest that there will always be at least one solution. And as $x$ tends to infinity, so does $x^{3}+b x^{2}+c x+d$, because for large $x, x^{3}$ is much bigger than the absolute values of any of the other terms, as well as their sum. Similarly, as $x$ tends to negative infinity, so does $x^{3}+b x^{2}+c x+d$. Therefore, the graph of the cubic must cross the horizontal axis somewhere, and where it crosses corresponds to a real-valued solution to the cubic equation. This is different from quadratic functions, whose graphs may not cross the horizontal axis at all.

So, let $r$ be a real-valued solution to the cubic equation.
If I long divide $x^{3}+b x^{2}+c x+d$ by $x-r$, I will get a quadratic and a constant remainder, so I know I can always find an identity that looks like

$$
x^{3}+b x^{2}+c x+d=(x-r)\left(x^{2}+m x+n\right)+C
$$

for some constants $m, n$, and $C$. But since $r$ is a solution to the cubic equation, it must be that $C=0$. So our cubic can be factored as a product of $x-r$ and a quadratic; I know how to find the roots of a quadratic, so if I can find one real solution of the cubic, I'll be able to find all of its roots.

But how can I even find one real root of the cubic?
I wonder if it would be easier to find one real root if I knew that there was precisely one real root, because then the real root would be unique, and it might be easier to find the real root as opposed to $a$ real root. I'll try that.

I know that one circumstance in which there could be only one real root is if the cubic is strictly increasing with $x$, and that would happen if the derivative of the cubic with respect to $x$ is always positive. The derivative of the cubic is

$$
3 x^{2}+2 b x+c
$$

and this is positive for all $x$ if its discriminant, $4 b^{2}-12 c$, is negative; we know this because if the discriminant is negative, the derivative doesn't have real roots, so cannot cross the horizontal axis.

So let's assume that $4 b^{2}-12 c<0$, or $b^{2}<3 c$.
Since complex roots come in conjugate pairs, the roots of the cubic would have to be $r, p+q i$, and $p-q i$, where $i$ is the square root of -1 and $q>0$.

I don't know if this idea will lead anywhere, but if I translate the cubic horizontally by $-p$, its two non-real roots would then be on the imaginary axis, and the quadratic equation with those roots has the form $x^{2}+q^{2}$. That is, there must be some $t$ such that translation of the cubic horizontally by $t$ must result in a cubic of the form $(x-s)\left(x^{2}+q^{2}\right)$, and only one value of $t$ should yield a cubic that can be so factored (because, for any other value, the complex roots would not be on the imaginary axis). This means that there should be a unique value of $t$ such that

$$
(x-t)^{3}+b(x-t)^{2}+c(x-t)+d=(x-s)\left(x^{2}+q^{2}\right)
$$

for some (real numbers) $s$ and $q$.
Expanding and gathering like terms, this identity becomes

$$
x^{3}+(b-3 t) x^{2}+\left(c-2 t b+3 t^{2}\right) x+\left(-t^{3}+b t^{2}-c t+d\right)=x^{3}-s x^{2}+q^{2} x-s q^{2} .
$$

Equating coefficients, we find that

$$
\begin{gathered}
s=3 t-b \\
q^{2}=c-2 t b+3 t^{2} \\
s q^{2}=t^{3}-b t^{2}+c t-d
\end{gathered}
$$

Hey, the third equation is the product of the first two! This means that

$$
(3 t-b)\left(3 t^{2}-2 t b+c\right)=t^{3}-b t^{2}+c t-d .
$$

Simplifying, I get

$$
8 t^{3}-8 b t^{2}+2\left(c+b^{2}\right) t+d=0
$$

Drat! Maybe I should have known: to find this special value $t$, I need to solve another cubic equation! So, that approach doesn't help, unless this cubic equation happens to be one I can solve by using something like the rational root theorem. But that doesn't seem likely to happen in general.

Hm. I know this is a detour, but only one value of $t$ could possibly be the appropriate value to shift the graph horizontally and result in the two nonreal roots being on the imaginary axis, so this last cubic equation must have a unique real root. Even though the earlier condition I found for having only one real root is not a necessary condition, I' m curious if this cubic also satisfies it. First, I must divide throughout by 8 to make the lead coefficient 1 :

$$
t^{3}-b t^{2}+\left(c+b^{2}\right) t / 4+d / 8=0
$$

Is it true that $(-b)^{2}<3\left(c+b^{2}\right) / 4$ ?
How about that! It, in fact, simplifies to exactly $b^{2}<3 c$, which is what I was assuming about the original cubic!

The unique real solution $t$ is the negative of the real part of the nonreal roots of the original cubic; so although this idea didn't lead me to a cubic formula, I now know that if

$$
x^{3}+b x^{2}+c x+d
$$

is a cubic with two nonreal roots, then the real part of the nonreal roots is the root of another cubic - the cubic $x^{3}+b x^{2}+\left(c+b^{2}\right) x / 4-d / 8$ (the sign changes come from setting $\left.x=-t\right)$, with coefficients that are rational functions of $b, c$, and $d$. I wonder if that points to a more general truth, namely, if I have a polynomial of degree $n$ with rational coefficients, will the real parts of its roots be, themselves, roots of a polynomial of degree $n$ with rational coefficients?

I'll have to think about that later. Back to trying to find a cubic formula!
What else can I try?

# Meditate ${ }^{\text {Math }}$ 

by Addie Summer
The topic of this Meditate to the Math is Descartes' Rule of Signs:

Consider a polynomial with real coefficients. Let $S$ be the number of changes of sign in the coefficients as you go through them in order of increasing degree. Let $R$ be the number of positive real roots of the polynomial (counted with multiplicity). Then $R \leq S$ and $R=S(\bmod 2)$.

Find a quiet place to think. Try to think your way to understanding the truth of Descartes' rule. If you get stuck, thinking about one of the statements below might help.

If $S=0$, can you see that there are no positive real roots (i.e., $R=0$ )?

What happens when the polynomial only has two terms? Three terms?

Suppose the first and last (in order of degree) nonzero coefficients have the same sign. Is $S$ even or odd?

Without loss of generality, you can assume that the constant term of the polynomial is not zero.

Let $r$ be a root of the polynomial with multiplicity $m$. How does the sign of the polynomial change as you cross over this root $r$ ?

> What can you say about the positive real roots of the derivative of the polynomial?

## Notes from the Club

These notes cover some of what happened at Girls’ Angle meets. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than $5 \%$ of what happens at the club is revealed here.

Session 34 - Meet 1 Mentors: Elisabeth Bullock, Serina Hu, Shauna Kwag,
February 1, 2024
Gautami Mudaliar, Hanna Mularczyk, AnaMaria Perez, Padmasini Venkat, Jane Wang, Doris Woodruff, Saba Zerefa

One of our fifth grade members proved that for any positive integer $n$, there exists a Fibonacci number which is a multiple of $n$. Can you prove this too?

Session 34 - Meet 2 Mentors: Elisabeth Bullock, Jade Buckwalter, Anushree Gupta, February 8, 2024 Clarise Han, Serina Hu, Gautami Mudaliar, Hanna Mularczyk, AnaMaria Perez, Vievie Romanelli, Swathi Senthil, Padmasini Venkat

Let $n$ be a positive integer and let $S(n)$ be the sum of the divisors of $n$. Can you prove that $S(a b)=S(a) S(b)$ when $a$ and $b$ are relatively prime positive integers?

There's a wealth of mathematics in sorting algorithms. How many algorithms can you think of that sort a list according to some criterion? For each algorithm you come up with, in what order should the elements be for the worst-case scenario, i.e., the scenario where the algorithm would take the longest to sort out the list? How long would you expect your algorithm to take to sort a list on average?

Session 34 - Meet 3 Mentors: Elisabeth Bullock, Jade Buckwalter, Anushree Gupta, February 15, 2024

Clarise Han, Gautami Mudaliar, Hanna Mularczyk, AnaMaria Perez, Padmasini Venkat, Jing Wang, Dora Woodruff, Saba Zerefa

Suppose you have two polynomials $p(x)$ and $q(x)$. How can you find the sum and the product of the roots that are common to both polynomials?

Session 34 - Meet 4 Mentors: Elisabeth Bullock, Jade Buckwalter, Clarise Han, February 29, 2024 Shauna Kwag, Gautami Mudaliar, Hanna Mularczyk, AnaMaria Perez, Swathi Senthil, Dora Woodruff, Saba Zerefa

Through the years, members have invented a number of games that involve math. It happened again, at this meet! What math game might you create? To be clear, here, I do not mean using math to analyze the game, but that math is a main subject of the game. Perhaps the members who created the game will refine it to a point where we'll describe it in this magazine. We'll see!

## Calendar

Session 33: (all dates in 2023)
September 14 Start of the thirty-third session!
21
28 Support Network Visitor: Isable Vogt, Brown University
October 5
12
19
26
November 2
9
16
23 Thanksgiving - No meet
30
December 7
Session 34: (all dates in 2024)

| February | 1 | Start of the thirty-fourth session! |
| :--- | :---: | :--- |
|  | 8 |  |
|  | 15 |  |
|  | 22 | No meet |
| March | 29 |  |
|  | 7 |  |
|  | 14 |  |
|  | 21 |  |
| April | 28 | No meet |
|  | 4 |  |
|  | 11 |  |
|  | 18 | No meet |
| May | 25 |  |
|  | 2 |  |
|  | 9 |  |

Girls' Angle has run over 150 Math Collaborations. Math Collaborations are fun, fully collaborative, math events that can be adapted to a variety of group sizes and skill levels. We now have versions where all can participate remotely. We have now run four such "all-virtual" Math Collaboration. If interested, contact us at girlsangle@gmail.com. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls’ Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

## Girls’ Angle: A Math Club for Girls <br> Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) $\qquad$ (first) $\qquad$
Parents/Guardians: $\qquad$
Address (the Bulletin will be sent to this address):

Email:

Home Phone: $\qquad$ Cell Phone: $\qquad$
Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

The $\$ 50$ rate is for US postal addresses only. For international rates, contact us before applying.
Please check all that apply:Enclosed is a check for $\$ 50$ for a 1-year Girls' Angle Membership.I am making a tax-free donation.
Please make check payable to: Girls’ Angle. Mail to: Girls’ Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle @gmail.com.


A Math Club for Girls

## Girls’ Angle Club Enrollment

## Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls’ Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls’ Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome all girls (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.
How do I pay? The cost is $\$ 20 /$ meet for members and $\$ 30 /$ meet for nonmembers. Members get an additional $10 \%$ discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls’ Angle located? Girls’Angle is based in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to Girls’ Angle and send to Girls’ Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls’Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, founder and director of the Exploratory

Yaim Cooper, Institute for Advanced Study
Julia Elisenda Grigsby, professor of mathematics, Boston College
Kay Kirkpatrick, associate professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, assistant dean and director teaching \& learning, Stanford University
Lauren McGough, postdoctoral fellow, University of Chicago
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, associate professor, University of Utah School of Medicine
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Liz Simon, graduate student, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, associate professor, University of Washington
Karen Willcox, Director, Oden Institute for Computational Engineering and Sciences, UT Austin Lauren Williams, professor of mathematics, Harvard University

At Girls’ Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high-level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle any field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

## Girls’ Angle: Club Enrollment Form

Applicant's Name: (last) $\qquad$ (first) $\qquad$
Parents/Guardians: $\qquad$
Address: $\qquad$ Zip Code: $\qquad$
Home Phone: $\qquad$ Cell Phone: $\qquad$ Email: $\qquad$

Please fill out the information in this box.
Emergency contact name and number: $\qquad$

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? Yes No

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

Date: $\qquad$
(Parent/Guardian Signature)
Participant Signature:
Members: Please choose one.
$\square$ Enclosed is $\$ 216$ for one session (12 meets)I will pay on a per meet basis at $\$ 20 /$ meet.

> Nonmembers: Please choose one. $$
\quad \text { I will pay on a per meet basis at } \$ 30 / \text { meet. }
$$ $\square \quad$ I'm including $\$ 50$ to become a member, and I have selected an item from the left.

I am making a tax-free donation.

Please make check payable to: Girls' Angle. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

## Girls' Angle: A Math Club for Girls <br> Liability Waiver

I, the undersigned parent or guardian of the following minor(s)
do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.
$\qquad$ Date: $\qquad$ Print name of applicant/parent: $\qquad$
Print name(s) of child(ren) in program: $\qquad$


[^0]:    On the cover: Cubic Celebration by
    C Kenneth Fan. Foreground made with
    MATLAB, a product of MathWorks.
    Background courtesy of NASA.

[^1]:    ${ }^{1}$ Both authors are from the Department of Mathematical Sciences at the University of Wisconsin, Milwaukee.
    ${ }^{2}$ This publication supported in part by a grant from MathWorks.
    ${ }^{3}$ In some cases, edges are taken to be ordered pairs of vertices and can be thought of as an arrow that points from one vertex to another. Notice that in our definition, an edge cannot connect a vertex to itself.

[^2]:    ${ }^{4}$ For those new to chess, we recommend this primer: www.buffalolib.org/sites/default/files/gamingunplugged/inst/1\%20Basic\%20Chess\%20Instructions.pdf.

[^3]:    ${ }^{5}$ If you are curious to learn more about recent results in this area, we encourage you to search www.arxiv.org, which is a repository of preprints in mathematics, physics, and computer science.

[^4]:    ${ }^{6}$ This content is supported in part by a grant from MathWorks.

