## Girlsfe Bulletin <br> August/September 2023 • Volume 16 • Number 6

To Foster and Nurture Girls' Interest in Mathematics



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## From the Founder

If we compare how math is handled in school to how professionals handle math, two entirely different worlds emerge. One is a world of grades, rote learning, and contests. The other is a world of publications, understanding, and creativity...a strange discrepancy! - Ken Fan, President and Founder


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## Girls' Angle: A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

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[^0]
## An Interview with Sarah Spence Adams, Part 2

This is the concluding half of our interview with Sarah Spence Adams, Professor of Mathematics and Electrical \& Computer Engineering at Olin College.

Ken: Could you please explain one of your own achievements related to wireless communications?

Sarah: I have a funny story about working with undergraduates to solve a big problem related to space-time block codes, which are special matrices used to improve the reliability of wireless communications. When I was a new professor back in the early 2000's, I was introduced to these codes while working in Australia with Professor Jennifer Seberry. (As a relevant aside for Girls’ Angle members, I was able to travel to meet Jennie through the generosity of a mentoring travel grant provided by the Association for Women in Mathematics: always be on the look-out for funding opportunities for women in math!) These codes had recently appeared in the literature, so there were a lot of new, open problems that Professor Seberry and I began considering.

When I returned to teach at Olin College the next semester, I was obsessed with figuring out a solution to a certain major problem related to these new codes, which you can think of as special matrices of complex variables. I had two undergraduate students in my Discrete Mathematics class who were struggling to find a topic for their final project in my class, so I suggested they stop by my office to learn about space-time block codes and maybe do something in that area for their project. When I showed them what I was working on, they almost fainted! I had been

> Being a mathematician has given
> me the thrill of conquering
> unsolved problems, the
> camaraderie of working on teams
> in academia and in industry, and
> the honor of mentoring the next
> generation of problem solvers.

working to identify various substructures inside the matrices by using photocopied examples from published papers, highlighting different complex variables with different colors of markers, cutting the rows of the matrices into tiny strips of paper, rearranging the strips of paper into new patterns, taping them onto new sheets of paper, etc. My students, who were both avid programmers, were aghast at what I had been doing by hand! They told me that they could write some code to automate my tasks with simple clicks of a button, and that was the start of a multi-year collaboration that included answering the big question in the field and then branching out to other related questions.

I love this story because it was such a thrilling time in my research career, racing to answer this big question while I was a new professor working with early undergraduates, and it is such a poignant example of how people from different backgrounds can join together as complementary partners. Sometimes I would have an idea that these students could implement in an hour rather than the month it would have taken me by hand, and sometimes they would create new software that illuminated a pattern that I could then generalize and prove due to my training in pure math. We shared many laughs over the years at how differently we would approach everything, but it really was a dream team. As a student, I was often told about the power of collaboration, but this experience really drove home this lesson for me.

Ken: When did you first become interested in math and what sparked your interest?

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We're committed to producing quality math educational content and make every effort to provide this content to you for free.

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## Tennis: By The Numbers

by Tara Gall I edited by Amanda Galtman

The author made this investigation during her senior year at Clearview Regional High School as part of a special math course created by Anne Paoletti.

## Introduction

In this paper, we analyze the probabilities of winning a singles tennis game depending on the score and skill level difference between the players. We generalize our results to modified tennis games that are won by scoring $n$ points while winning by 2 . (We will deviate from proper tennis lingo in places. For proper lingo, we suggest consulting the United States Tennis Association.)

## Tennis Scoring

Most tennis matches are organized into sets, which consist of a series of games, each worth one point. A game is decided by rally scoring. To win, a player must score at least four points by a margin of two. In singles tennis, one player serves throughout the entire game. We call the players Player 1 and Player 2. We write the score as $A-B$, where $A$ is Player 1's score and $B$ is Player 2's score. This tree diagram shows all possible score situations and how they can progress. Technically, if Player 2 wins the point from a score of 3-2, the score becomes 3-3, but the game situation of 3-3 is logically equivalent to 2-2. The game continues until it reaches a "Win" or "Lose" state. This diagram will help us to compute win probabilities.


Figure 1. A tree diagram of all point situations in a tennis game with outcomes stated from the perspective of Player 1 as server. (Tennis lingo dubs these scores differently, but this diagram is logically equivalent to tennis scoring.)

## Modeling Tennis

To model the tremendous range of skill levels in tennis, we introduce the variable $L$ (for "level"), which we define to be the probability that Player 1 wins a single rally. Note that Player 2's skill level is $1-L$. For example, consider the 2023 French Open women's final, with Iga Świątek (a math lover!) as Player 1 and Karolína Muchová as Player 2. When Świątek was serving, her empirical value of $L$ was 51/88; when receiving, $L$ was 45/89.

In our model, we assume that each rally has an outcome that is independent of the other rallies. Thus, at each stage of the game, we multiply by that player's skill level to determine the probability that the game proceeds with that player winning the point. Please refer to Figure 2, which gives the probabilities that a tennis game sees each score shown, starting from 0-0. All games start at $0-0$, so we assign that score a probability of 1 . For the probability of score 1-0, we multiply 1 by $L$ since Player 1 wins the point with probability $L$. For score $0-1$, we multiply 1 by $1-L$ since Player 2 wins the point with probability $1-L$. At the top, the diagram continues to where Player 1 wins the first two points of the game. Here, we multiply $L$ by $L$ to get $L^{2}$. When Player 1


Figure 2. A tree diagram showing the probabilities that certain scores occur starting from a score of 0-0. wins the first point and Player 2 the second, we multiply $L$ by $1-L$, which is the same probability as when Player 2 wins the first point and then Player 1 wins the second. These probabilities are the same because the probability of reaching a particular score does not depend on the order in which the points are scored.

Because a player must win by two, it is conceivable that the score will continue to tie. However, it turns out that the never-ending game is a probability-zero event.

Since these games can, in theory, go on forever, we might sum an infinite series to compute the probability of winning. We can avoid infinite sums in the following way: we define the variable $T$ as the probability that Player 1 wins from a tie score of $k-k$, with $k \geq 2$. When $k$ is greater than or equal to two, all such scores are identical situations in that the next two rallies lead to either a win or another tie. By contrast, a winner of the next two points from $0-0$ or 1-1 has not yet won the game.

We can figure out how $T$ depends on $L$ by computing the probability that Player 1 wins from a score of 2-2 in two different ways. The first way is to use the definition of $T$ as the probability that Player 1 wins from a score of 2-2. The second way is to look at a tree diagram that starts at the score 2-2 in Figure 3.


Figure 3. Modeling the "deuce" situation in tennis.

By "playing out" the game from a score of 2-2 at least two more points until a win or loss occurs, we can break down the probability of Player 1 winning from a score of 2-2 as

$$
L^{2}+T(2 L(1-L)),
$$

which is the sum of the probabilities of the two mutually exclusive win nodes in Figure 3. Thus, we have the equation

$$
T=L^{2}+T(2 L(1-L))
$$

Solving for $T$, we find

$$
T=\frac{L^{2}}{1-2 L(1-L)} .
$$

Using the full tree diagram and using $T$ where appropriate, we find that the probability that Player 1 wins a tennis game is

$$
L^{4} \frac{15-34 L+28 L^{2}-8 L^{3}}{1-2 L(1-L)}
$$

In a similar manner, we computed the probability that Player 1 wins from any score situation reached during a tennis game. The results are in Table 1.

| $\boldsymbol{B} \boldsymbol{A}$ | $\mathbf{4}$ | $\mathbf{3}$ | $\mathbf{2}$ | $\mathbf{1}$ | $\mathbf{0}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathbf{4}$ | $\frac{L^{2}}{1-2 L(1-L)}$ | $\frac{L^{3}}{1-2 L(1-L)}$ | 0 | 0 | 0 |
| $\mathbf{3}$ | $\frac{L-L^{2}+L^{3}}{1-2 L(1-L)}$ | $\frac{L^{2}}{1-2 L(1-L)}$ | $\frac{L^{3}}{1-2 L(1-L)}$ | $\frac{L^{4}}{1-2 L(1-L)}$ | $\frac{L^{5}}{1-2 L(1-L)}$ |
| $\mathbf{2}$ | 1 | $L \frac{1-L+L^{2}}{1-2 L(1-L)}$ | $\frac{L^{2}}{1-2 L(1-L)}$ | $L^{3} \frac{1+L-2 L^{2}}{1-2 L(1-L)}$ | $L^{4} \frac{1+2 L-2 L^{2}}{1-2 L(1-L)}$ |
| $\mathbf{1}$ | 1 | $L \frac{2-4 L+4 L^{2}-L^{3}}{1-2 L(1-L)}$ | $L^{2} \frac{3-5 L+4 L^{2}-L^{3}}{1-2 L(1-L)}$ | $L^{3} \frac{4-5 L+2 L^{2}}{1-2 L(1-L)}$ | $L^{L^{4} \frac{5-4 L-2 L^{2}+2 L^{3}}{1-2 L(1-L)}}$ |
| $\mathbf{0}$ | 1 | $\frac{L\left(3-8 L+10 L^{2}-5 L^{3}+L^{4}\right)}{1-2 L(1-L)}$ | $L^{2} \frac{6-16 L+19 L^{2}-10 L^{3}+2 L^{4}}{1-2 L(1-L)}$ | $L^{3} \frac{10-25 L+20 L^{2}-12 L^{3}+2 L^{4}}{1-2 L(1-L)}$ | $L^{4} \frac{15-34 L+28 L^{2}-8 L^{3}}{1-2 L(1-L)}$ |

Table 1. The probability that Player 1 wins from a score of $A-B$.
This chart helps us visualize how complicated a game of tennis can be. When the score is lower, the probability expression is more complicated, reflecting the many different ways the game can play out. As the score gets higher, these expressions become less complicated since there are fewer unique outcomes to account for.

As an example, suppose Player 1 wins rallies $75 \%$ of the time. What is the probability that Player 1 wins from a score of 3-2? From the table, the probability is

$$
L \frac{1-L+L^{2}}{1-2 L(1-L)}
$$

We substitute 0.75 for $L: 0.75\left(1-0.75+(0.75)^{2}\right) /(1-2(0.75)(1-0.75))=0.975$. Therefore, when the score is 3-2, Player 1 has a $97.5 \%$ chance of winning the entire game. Another fact we can deduce from this formula is that if the score is 3-2, Player 1 still has a better than 50-50 chance of winning the game even if Player 1 wins only about $35.22 \%$ of the rallies. To find this, we set the above expression equal to $1 / 2$ and solve for $L$.

Back at the 2023 French Open women's final, with $L=51 / 88$, our formula predicts that Świątek would win $69 \%$ of the games where she served. In fact, she won 10 out of 15 , or about $67 \%$.

## Generalization

We can imagine a game just like the tennis game, except requiring that players achieve a higher score than four (by two) in order to win. However, computing the probability of Player 1 winning from various scores using the technique we applied to the tennis game requires a lot of tedious computation. Is there a more efficient way to analyze a tennis game where players have to reach $N$ points (by two), instead of four?

For tennis, tie scores $k-k$ for $k \geq 2$ correspond to identical situations: the next two points lead to either a win or another tie. But if the game goes to $N$ points, then the tie scores $k-k$ become identical only for $k \geq N-2$. So, it makes sense to now define $T$ to be the probability that Player 1 wins the game if the score is $k-k$ for $k \geq N-2$. From such scores, the situation becomes identical to the tennis game at a score of 2-2. Therefore, we still have $T=L^{2} /(1-2 L(1-L))$. In fact, the probability of winning from $A-B$ with $A \geq N-4$ and $B \geq N-4$ is the same as the probability we found for the tennis game for the score $(A-N+4)-(B-N+4)$. But what about the other scores?

Let's define $p_{N}(a, b)$ to be the probability that Player 1 wins the game from a score of $a-b$ in a tennis-like game where, to win, a player must reach $N$ points and be ahead by two.

How can we compute $p_{N}(a, b)$ ?
Only two things can happen with the next point: either Player 1 wins it or Player 2 wins it. Therefore,

$$
p_{N}(a, b)=L p_{N}(a+1, b)+(1-L) p_{N}(a, b+1) .
$$

It may seem like we can use this equation to solve for any score in the game now, but we cannot. This is a recurrence relation, and one must know at least some values from which to begin the recursion in order to use a recurrence relation to compute specific values. We do know that $p_{N}(N, k)=1$ and $p_{N}(k, N)=0$ for $k \leq N-2$. And we also know (from Table 1 ) the values of $p(N, N-1), p(N, N)$, and $p(N-1, N)$. From these values, one could use the recursion formula to find all the probabilities.

This process is less tedious than computing the probabilities from each score by following the tree of scoring possibilities, but there are more patterns that help simplify the computations.

Note that almost all the expressions in Table 1 have a numerator that looks like some power of $L$ multiplied by a polynomial in $L$. In fact, the numerators all have a factor of $L^{N-a}$. Also, the denominators for the expressions are all the same: $1-2 L(1-L)$. This suggests that $p_{N}(a, b)$ has the form

$$
p_{N}(a, b)=L^{N-a} \frac{q_{N}(a, b)}{1-2 L(1-L)},
$$

where $q_{N}(a, b)$ is a polynomial in $L$. To see that this is justified, we substitute this expression into our recursion formula:

$$
\begin{aligned}
p_{N}(a, b) & =L p_{N}(a+1, b)+(1-L) p_{N}(a, b+1) \\
& =L\left(L^{N-(a+1)} \frac{q_{N}(a+1, b)}{1-2 L(1-L)}\right)+(1-L)\left(L^{N-a} \frac{q_{N}(a, b+1)}{1-2 L(1-L)}\right) \\
& =\frac{L^{N-a}}{1-2 L(1-L)}\left(q_{N}(a+1, b)+(1-L) q_{N}(a, b+1)\right) .
\end{aligned}
$$

Thus,

$$
\frac{L^{N-a}}{1-2 L(1-L)} q_{N}(a, b)=\frac{L^{N-a}}{1-2 L(1-L)}\left(q_{N}(a+1, b)+(1-L) q_{N}(a, b+1)\right) .
$$

Simplifying, we find that

$$
q_{N}(a, b)=q_{N}(a+1, b)+(1-L) q_{N}(a, b+1) .
$$

Using the known values for $p_{N}(a, b)$ when $a=N$ or $b=N$, we find that

$$
\begin{aligned}
q_{N}(N, k) & =1-2 L(1-L) & & \text { for } k \leq N-2, \\
q_{N}(k, N) & =0 & & \text { for } k \leq N-2, \\
q_{N}(N, N-1) & =L\left(1-L+L^{2}\right) & & \\
q_{N}(N, N) & =L^{2} & & \\
\text { and } q_{N}(N-1, N) & =L^{2} . & &
\end{aligned}
$$

Since $q_{N}(a, b)$ can be computed recursively from the above expressions, we see that all the $q_{N}(a, b)$ are, in fact, polynomials in $L$ and our observation is justified.

In other words, we have reduced our computations to working with polynomials instead of rational functions!

## Functional Relation

Notice that $p_{N}(a, b)(1-L)$ (i.e., substituting $1-L$ for $L \ldots$ in this subsection, the parenthetical expressions after $p_{N}(a, b)$ or $p_{N}(b, a)$ express functional dependency and not multiplication) is the same as the probability of Player 2 winning from a score of $a-b$. Now, the probability of Player 2 winning from a score of $a-b$ is 1 minus the probability of Player 1 winning from a score of $b-a$. That is,

$$
p_{N}(a, b)(1-L)=1-p_{N}(b, a)(L) .
$$

## Further Investigation

We saw that the numerators of our probabilities $p_{N}(a, b)$ have a factor of $L^{N-a}$. What causes this behavior? Is there a non-recursive formula for the probabilities $p_{N}(a, b)$ (or the polynomials $\left.q_{N}(a, b)\right)$ ? If you figure out any of these or related questions, please email us at girlsangle@gmail.com!

## A Self-Referential Multiple Choice Test

## by Sascha McHugh

To ace this test, you don't need to have any specific knowledge. You just have to be at your logical best!


1 The only question with the answer $C$ is...
(A) 1
(B) 2
(C) 3
(D) 4
(E) 5

2 The answer to the following question is...
(A) E
(B) D
(C) C
(D) B
(E) A

3 The next question with the same answer as this one is...
(A) 2
(B) 5
(C) 1
(D) 3
(E) 4

4 The number of questions with a vowel as an answer is..
(A) 5
(B) 2
(C) 4
(D) 3
(E) 1

5 The number of questions before this with the same answer as this one is...
(A) 2
(B) 1
(C) 0
(D) 3
(E) 4

6 The first question with the answer (A) is...
(A) 6
(B) 1
(C) 3
(D) 9
(E) 5

7 The number of questions with the answer (D) is...
(A) 4
(B) 2
(C) 5
(D) 1
(E) 3

8 The only consecutive questions with the same answers are...
(A) 1 and 2
(B) 3 and 4
(C) 9 and 10
(D) 7 and 8
(E) 2 and 3

9 The question before this with the answer (E) is...
(A) 4 away
(B) 1 away
(C) 0 away
(D) 2 away
(E) 3 away

10 Alphabetically, the answer to question 6 is $\qquad$ letters away from this answer.
(A) 15
(B) 7
(C) 18
(D) 6
(E) 4

If you enjoyed this, there's another created by Ghost Inthehouse, HolAnnherKat, Katnis
Everdeen, and Shark Inthepool on pages 20-21 of Volume 11, Number 2 of this Bulletin and another by Michelle Chen on page 13 of Volume 13, Number 2. Also, check out the one that inspired these, which was written by James Propp and can be found on the internet.

## A Pascal's Triangle for Partition Numbers

by Robert Donley ${ }^{1}$

edited by Amanda Galtman
In the previous installment, we used partial sums, partial fractions, and the binomial series to find interesting properties of partition functions. In this part, we draw connections between partition numbers and Pascal's triangle, an object that we have continually revisited throughout this series. Some problems require accessing the On-Line Encyclopedia of Integer Sequences (OEIS) to identify sequences of partition numbers. See oeis.org.

Definition. A partition of a positive integer $k$ into $m$ parts is a sum of entries $x_{1}+\ldots+x_{m}=k$ where each $x_{i}$ is a positive integer and $x_{i} \geq x_{i+1}$. (The $x_{i}$ are the parts of the partition and $m$ is the length of the partition.)

When all parts are single-digit numbers, we simply list the parts as a string of digits.
We define the function $p(k)$ to be the number of partitions of $k$ of any length, while $p_{[n]}(k)$ denotes the number of partitions of $k$ with parts in $1, \ldots, n$. Of special interest is the function $p_{n}(k)$, the number of partitions of $k$ with largest part $n$. As a convention, we define $p(0)=1$, and likewise for other partition functions used here.

Recall Euler's generating function for the partition function $p(k)$ from the previous installment:

$$
F(t)=p(0)+p(1) t+p(2) t^{2}+p(3) t^{3}+\ldots=\frac{1}{1-t} \frac{1}{1-t^{2}} \frac{1}{1-t^{3}} \cdots .
$$

To find a simpler but still useful model to direct our work, let's see what happens if we remove the exponents from the rightmost expression. Now we have the generating function

$$
P(t)=\frac{1}{1-t} \frac{1}{1-t} \frac{1}{1-t} \cdots .
$$

Term-wise, this generating functions models iterations of partial sums, a familiar scenario with many useful properties. The leading $n+1$ factors of $P(t)$ give the binomial series

$$
\frac{1}{(1-t)^{n+1}}=1+\binom{n+1}{1} t+\binom{n+2}{2} t^{2}+\ldots
$$

The presence of binomial coefficients suggests that patterns in Pascal's triangle might have analogues in the study of partitions. Let's find out!

Exercise: As an alternative to Pascal's identity, construct Pascal's triangle using partial sums along diagonals, starting with the rightmost diagonal of 1 s .

[^1]This construction begins by recording the coefficients of the geometric series $1 /(1-t)$ as the rightmost diagonal of 1 s , and each multiplication by $1 /(1-t)$ implements the partial sum operation. This construction basically consists of repeated applications of the hockey stick rule.

Exercise: Skim through the previous nine installments and note any instances of binomial coefficients or Pascal's triangle. Of particular interest are any binomial coefficient identities that are expressed by diagrams in Pascal's triangle.

## Warm-up exercises and finite differences

For the following exercises, find the corresponding entry in the OEIS. If you use generating functions to obtain values, check your sequence also by listing the partitions for the first few terms.

Exercise: Fix a positive integer $n$. Prove that the sequence that counts partitions of $k$ with $n$ or fewer parts is the same as the sequence for $p_{[n]}(k)$. Verify by listing the corresponding partitions when $n=1,2,3$. Alternatively, explain why these sequences have the same generating function.

Exercise: Prove that $p_{n}(k)$ is also the number of partitions of $k$ with exactly $n$ parts. What is the relationship between $p_{n}(k)$ and $p_{[n]}(k)$ ?

Exercise: Find the sequence of partition numbers with odd numbers for parts. For instance, $p_{\text {odd }}(5)=3$ with partitions 5,311 , and 11111. Find the generating function, and, by applying the identity

$$
\frac{1-t^{2 k}}{1-t^{k}}=1+t^{k}
$$

give a second form of the generating function and another interpretation of the sequence.
Exercise: Find the sequence of partition numbers for two or more parts. For instance, $p_{>1}(5)=6$ with partitions $41,32,311,221,2111$, and 11111 . Find a simple formula for $p_{>1}(k)$ in terms of $p(k)$.

Exercise: Find the sequence of partition numbers using only parts greater than 1. For instance, $p_{[>1]}(5)=2$ with partitions 5 and 32 . By manipulating generation functions or by direct argument, prove that

$$
p_{[>1]}(k)=p(k)-p(k-1) .
$$

The previous exercise indicates a useful technique for calculating partition numbers when parts are removed. The finite difference operator $D$ undoes the operation of partial sums, and vice versa. Let $a_{n}, n \geq 0$, be a sequence of real numbers. Define

$$
S a(k)=a_{0}+\ldots+a_{k} \text { and } D a(k)=a_{k}-a_{k-1} .
$$

We define $D a(0)=a_{0}$.
Exercise: Prove that, if $G(t)$ is the generating function for $a_{k}$, then the generating function for $D a(k)$ is $(1-t) G(t)$. What happens if we define $D_{n} a(k)=a_{k}-a_{k-n}$ when $k \geq n$ and $D_{n} a(k)=a_{k}$ otherwise?

Exercise: Verify that

$$
D(S a)(k)=a_{k} \text { and } D a(0)+D a(1)+D a(2)+\ldots+D a(k)=a_{k} .
$$

Express these identities in terms of the generating function $G(t)$ for $a_{k}$.
Exercise: Suppose $p_{A}(k)$ is the partition function using parts in some subset of positive integers $A$. What is the generating function for $p_{A}(k)$ ? Prove that, if $a$ is in $A$, then the partition function using parts in $A$ excluding $a$ is

$$
p_{A \backslash\{a\}}(k)=p_{A}(k)-p_{A}(k-a) .
$$

Use this formula to compute the closed formula for $p_{[2]}(k)$ from the closed formula for $p_{[3]}(k)$ given in the previous installment. If you found a formula for $p_{[4]}(k)$, compute $p_{[3]}(k)$ similarly.

Exercise: Construct a table for partition numbers formed from odd parts. First, imitate the construction of the main table for partition numbers from the previous installment using the equation

$$
O(t)=F(t)\left(1-t^{2}\right)\left(1-t^{4}\right)\left(1-t^{6}\right) \cdots
$$

where $O(t)$ is the generating function for the partition numbers using odd parts. The first row should be the partition numbers, and successive rows are calculated using aerated finite differences. Interpret each row of the table.

In the main table, finite differences allow us to zero out the repeated entries without losing information. To implement finite differences in the main table, we subtract from each entry the one above it. Surprisingly, we recover the original table, but it is now preceded by a triangle of zeros.

| $\boldsymbol{n} \boldsymbol{k}$ | $\mathbf{0}$ | $\mathbf{1}$ | $\mathbf{2}$ | $\mathbf{3}$ | $\mathbf{4}$ | $\mathbf{5}$ | $\mathbf{6}$ | $\mathbf{7}$ | $\mathbf{8}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathbf{1}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\mathbf{2}$ | 0 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 |
| $\mathbf{3}$ | 0 | 0 | 0 | 1 | 1 | 2 | 3 | 4 | 5 |
| $\mathbf{4}$ | 0 | 0 | 0 | 0 | 1 | 1 | 2 | 3 | 5 |

Exercise: Interpret the following equalities in terms of of the above table:

$$
p_{n}(k)=p_{[n]}(k)-p_{[n-1]}(k)=p_{[n]}(k-n) .
$$

Also prove the equalities by using generating functions.

## Main task: A Pascal's triangle for partitions (OEIS A008284 reversed along rows)

In the previous installment, we constructed the table for partition numbers using aerated partial sums. To obtain the corresponding triangle of partition numbers, we rotate this table by $45^{\circ}$ clockwise, so that the top row of the table is now the rightmost diagonal. We denote the apex as
row 1 of the triangle in agreement with the row number in the main table. Please add more rows as needed.

We first note that, unlike Pascal's triangle, this triangle is not symmetric about the central vertical line, so each property that we find has both left and right versions.


Left and right diagonals: In Pascal's triangle, the diagonals form coefficients of binomial series. If we identify the diagonals of the partition triangle as rows and columns of the main table, then the diagonals to the lower right are sequences of partition numbers $p_{[n]}(k)$ for fixed $n$. On the other hand, the diagonals to the lower left are partition numbers for fixed $k$.

Exercise: Explain why the values on diagonals to the lower left eventually stabilize, and determine where in the triangle this happens for each diagonal. Then explain why the vertical sequences to the left of the central vertical line are identical and equal to the sequence for $p(k)$.

Exercise: Explore figurate numbers. That is, enter the first few diagonals from Pascal's triangle into the OEIS to discover further applications of binomial series.

Exercise: In Pascal's triangle, the sum over the entries of the $n$th row is $2^{n}$. Identify the sequence of row sums in the partition triangle. Prove this result by considering partition numbers with fixed largest part. Repeat for alternating sums.

Pascal's identity/hockey stick rule: Along diagonals to the lower right, versions of Pascal's identity and the hockey stick rule are used to define the main table of partition numbers. To obtain the corresponding hockey stick rule in the other direction, we first emphasize the relationship between the two equations.

Exercise: Prove that the hockey stick rule and Pascal's identity are equivalent. On one hand, explain how the hockey stick rule follows from iterating Pascal's identity. On the other hand, express Pascal's identity as a difference of the entries in two hockey sticks.

Pascal's identity for the partition triangle is the three-term recurrence

$$
p_{[n]}(k)=p_{[n]}(k-n)+p_{[n-1]}(k),
$$

which we interpret as a splitting of the partitions of $k$ with parts in $\{1, \ldots, n\}$ into those with or without a part equal to $n$. The aerated partial sum formula from the previous installment gives one of the two hockey stick rules.

Exercise: Repeat the previous exercise using the three-term recurrence and the aerated partial sum formula for the main table. Draw diagrams for both equations at several nodes in the triangle. Interpret the aerated partial sum formula for $p_{[n]}(k)$ in terms of the part $n$.


Exercise: Prove that each entry of the partition triangle is less than or equal to the corresponding entry of Pascal's triangle. For the inequality $p_{[n]}(k) \leq\binom{ r}{s}$, find $r$ and $s$ in terms of $n$ and $k$. How many compositions of $k$ into $n$ parts are there?

Exercise: Prove that the other hockey stick rule in the partition triangle calculates partial sums of the row entries from the right-hand side. Interpret the partial sum by considering each term as counting partitions of $k$ with a given largest part. The row sum rule above is a special case of this hockey stick rule.


Diagonal sums: In the previous installment from Volume 16, Number 4, the Fibonacci numbers arise as sums over the entries of diagonals in the left- or right-justified Pascal's triangle. Consider both justifications of the partition triangle as pictured below.


Exercise: Find both sequences of diagonal sums in the OEIS. Then draw a general diagonal sum in each direction on the main table, and express the diagonal sum in terms of partition numbers.

For instance, for the diagonal in the left-justified triangle starting at row $n$, the sum matches $p_{\text {odd }}(n)$, with entry A000009 in the OEIS. Let's consider partitions of $n$ into odd parts. For instance, the partition 7311 of $n=12$ splits as $1111+6200$, and this representation is typical. The number of 1 s in the first term indicates the number of parts. The second term is always a partition with even parts but is restricted in number of parts by the first term.

Exercise: List all such partitions of 7, express the corresponding sums, and count these sums by the number of 1 s in the first term. Then construct all such partitions of 8 by reversing this process: for a given first term with an even number of 1 s , append all possible partitions of appropriate size having only even parts.

Exercise: Let $p_{\text {even }}(k)$ be the number of partitions of $k$ with even parts. (For example, $p_{\text {even }}(8)$ is 5: the partitions are $8,62,44,422,2222$.) Prove that $p_{\text {even }}(2 m)=p(m)$ and $p_{\text {even }}(2 m k+1)=0$.

Exercise: Explain why the sum of the counting formula for $p_{\text {odd }}(n)$ is given by

$$
p_{[n]}(0)+p_{[n-2]}(1)+p_{[n-4]}(2)+\ldots .
$$

Thus the counting formula for $p_{\text {odd }}(k)$ by number of parts agrees with the diagonal sum formula.
Exercise: Prove a similar equality for the diagonal sums of the right-justified triangle.
Finally, we obtain a version of Chu-Vandermonde convolution by splitting the generating function for $p(k)$. For sequences $a_{k}$ and $b_{k}$, the convolution is given by the sequence

$$
c_{k}=a_{0} b_{k}+a_{1} b_{k-1}+\ldots+a_{k} b_{0}
$$

the coefficient sequence for the product of the respective generating functions. Now $F(t)=E(t) O(t)$, where $E(t)$ and $O(t)$ are the generating functions corresponding to even and odd parts, respectively. If we apply the exercise for $p_{\text {even }}(k)$ to the convolution formula, we obtain

$$
p(k)=p(0) p_{\text {odd }}(k)+p(1) p_{\text {odd }}(k-2)+p(2) p_{\text {odd }}(k-4)+\ldots .
$$

Exercise: Verify the convolution formula for $5 \leq k \leq 10$. Interpret this formula as a variation on the formula for $p_{\text {odd }}(k)$.

For $r$ equal to 1 or 2 , let $p_{3, r}(k)$ be the number of partitions of $k$ with parts of the form $3 s+r$. Then $p(k)$ is the sum over all products of the form

$$
p(a) p_{3,1}(b) p_{3,2}(c)
$$

with $3 a+b+c=k$.

Exercise: Prove a general convolution formula for the product of three or more generating functions.

Exercise: Verify the threefold convolution formula for $p(k)$ with $3 \leq k \leq 5$. Interpret this formula as above. Do you see a general formula for partitions using parts with remainders upon division by $m$ ? Interpret as above.

## Members' Thoughts

## A Mathematical Gem: On a Theorem of Katherine Knox ${ }^{1}$ by Ken Fan

Take a polygon and imagine that its sides are mirrors. Is it possible for a laser beam to bounce off of each of its sides in clockwise order before closing up and looping around indefinitely?

This is the question that Katherine Knox dreamed up while in 6th grade. She dubbed a path such a beam would follow a light path. Katherine's theorem is about light paths in quadrilaterals, but she started her journey by analyzing light paths in triangles. Let's also begin with triangles to get a feel for this kind of mathematics.


Consider triangle $A B C$ with light path $D E F$. See the figure at left. Because we are talking about light bouncing off of a mirror, angles $F D B$ and $E D C$ have the same measure, which we label $a$ in the figure. Similarly, angles $D E C$ and $F E A$ are equal with measure $b$, and angles $E F A$ and $D F B$ are equal with measure $c$.

We use the convention that the label for a triangle's vertex is also the label of the measure of the angle at that vertex.

Following in Katherine's footsteps, we use the fact that the angles in a triangle sum to $180^{\circ}$ three times:

$$
\begin{aligned}
& A+b+c=180^{\circ} \\
& B+c+a=180^{\circ} \\
& C+a+b=180^{\circ}
\end{aligned}
$$

We can solve this system of equations for $a, b$, and $c$ in terms of the measures of the angles of the given triangle $A B C$. We get

$$
\begin{aligned}
& a=\left(180^{\circ}+A-B-C\right) / 2 \\
& b=\left(180^{\circ}+B-C-A\right) / 2 \\
& c=\left(180^{\circ}+C-A-B\right) / 2
\end{aligned}
$$

Since the angles of a triangle add up to $180^{\circ}$, we can substitute $A+B+C$ for $180^{\circ}$ in each of these equations to discover that $a=A, b=B$, and $c=C$ !

In other words, a light path will split the triangle into 4 triangles, a central one surrounded by 3 triangles that are each similar to the original triangle. The fact that $a=A, b=B$, and $c=C$ also shows that right and obtuse triangles cannot contain a light path, since the angles $a, b$, and $c$ must be acute.

Katherine also found the similarity ratios between the given triangle and each of the 3 smaller triangles, and showed that the light path will bounce on each side at the feet of the altitudes. All of this was known to the 18th century Italian mathematician Giovanni Fagnano.

[^2]Then, Katherine moved on to quadrilaterals.
Let $A B C D$ be a quadrilateral and suppose it contains a light path $E F G H$, as shown in the figure at right. Let $a, b, c$, and $d$ be the angle measures of the indicated angles, where the light path bounces upon the sides.

The equations for the quadrilateral analogous to the angle equations used to find the angles of incidence and reflection for the light path in a triangle are


$$
\begin{aligned}
& A+a+d=180^{\circ} \\
& B+b+a=180^{\circ} \\
& C+c+b=180^{\circ} \\
& D+d+c=180^{\circ}
\end{aligned}
$$

Unlike the triangle case, we have a problem: this system does not have a unique solution! What shall we do?

Well, if we add the first and third equations, we get $A+a+d+C+c+b=360^{\circ}$; and if we add the second and fourth equations, we get $B+b+a+D+d+c=360^{\circ}$. If we subtract the latter equation from the former, we get $A+C-B-D=0^{\circ}$. That means $A+C=B+D$. And since the sum of the angles in any quadrilateral is $360^{\circ}$, this tells us that $A+C=B+D=180^{\circ}$. In other words, to have a light path, the quadrilateral must be inscribable in a circle - a so-called cyclic quadrilateral. The necessity of having to be a cyclic quadrilateral was also known and appears in the comprehensive treatise Geometry and Billiards, by Serge Tabachnikov. But being cyclic does not guarantee a light path.

Because one cannot determine the angles $a, b, c$, and $d$ from the angles $A, B, C$, and $D$, it's harder to construct examples of light paths. Nevertheless, Katherine managed to find light paths in squares, rectangles, cyclic kites, and cyclic trapezoids. But ... not all cyclic trapezoids! For example, isosceles trapezoids with interior angles of $150^{\circ}$ and $30^{\circ}$ do not admit light paths. Which cyclic quadrilaterals have light paths? The answer to that, nobody knew!

Then came Katherine Knox.
"Ken!"
"Yes?"
"You know how it's the acute triangles that have light paths?"
"Yes ..."
"Well," she said, "the acute triangles are exactly the triangles that contain the centers of their circumcircles. So I conjecture that the quadrilaterals with light paths are exactly the cyclic ones that contain the centers of their circumcircles!"

And with that brilliant insight, Katherine Knox zapped the problem!
See the cover for a light path, and check out Katherine's proof in The American Mathematical Monthly: www.tandfonline.com/doi/full/10.1080/00029890.2023.2230860
(There, at the behest of one of the referees, "light path" became "billiard circuit.")

In the previous issue, we presented the 2023 Summer Fun problem sets.
In this issue, we give solutions to many of the problems. Our solutions may be terse and, in some cases, are more of a hint than a solution. We prefer not to give detailed solutions before we know that most of the members have spent time thinking about the problems. The reason is that doing mathematics is very important if you want to learn mathematics well. If you haven't tried to solve these problems yourself, you won't gain as much when you read these solutions.

If you haven't thought about the problems, we urge you to do so before reading the solutions. Even if you cannot solve a problem, you will benefit from trying. By working on the problem, you will force yourself to think about the associated ideas. You will gain familiarity with the related concepts and that will make it easier and more meaningful to read other's solutions.

With mathematics, don't be passive! Be active!
Move your pencil and move your mind - you might discover something new.

Also, the solutions presented are not definitive. Try to improve them or find different solutions.
Solutions that are especially terse will be indicated in red. Please do not get frustrated if you read a solution and have difficulty understanding it. If you run into difficulties, we are here to help! Just ask!

Please refer to the previous issue for the problems.

Members: Don't forget that you are more than welcome to email us with your questions and solutions!

## Best Ice Cream Cone by Clarise Han

1. There are $7 \times 7 \times 7=343$ possibilities.
2. There are $7 \times 6 \times 5=210$ possibilities.
3. There are $7 \times 6 \times 5 \times 4 \times 3=2,520$ possibilities.
4. Let's do casework on the number of scoops. Since Coco insists on having mint chocolate and cookie dough, there must be at least 2 scoops, so we begin enumerating cases starting with 2 scoops. Case 1: 2 scoops: 2 possibilities (because order matters!). Case 2: 3 scoops: 30 possibilities. Case 3: 4 scoops: 240 possibilities. Case 4: 5 scoops: 1,200 possibilities. In total, there are $2+30+240+1,200=1,472$ possibilities.
5. The total number of ice cream cones with five different scoops including mint chocolate and cookie dough is 1,200 . The number of ice cream cones with five different scoops and mint chocolate and cookie dough next to each other is 480 . Thus, the probability is $480 / 1,200$, or $2 / 5$.
6. There are ${ }_{9} C_{3}=(9 \times 8 \times 7) /(3 \times 2 \times 1)=84$ three-topping combinations. ${ }^{1}$
7. We perform complementary counting. The total number of five-topping combinations is ${ }_{9} C_{5}$, which is 126 . The number of five-topping combinations without hot fudge and caramel sauce is ${ }_{7} C_{5}=21$. Thus, the number of five-topping combinations with at least one of hot fudge and/or caramel sauce is $126-21=105$.
8. Let's do casework on the number of toppings. Case $1: 1$ topping: 1 topping combination. Case 2: 2 toppings: 8 topping combinations. Case $3: 3$ toppings: ${ }_{8} C_{2}=28$. Case $4: 4$ toppings: ${ }_{8} C_{3}=56$. Case 5: 5 toppings: ${ }_{8} C_{4}=70$. The number of ice cream cones with three different flavors including matcha is 90 . Thus, the total number of three-flavor ice cream cones with five toppings including matcha and popping boba is $90 \times(1+8+28+56+70)=14,670$.
9. The total number of ice cream cones with seven different flavors is 7 !. If we keep Coco's cone constant, the number of cones with all flavors in different spots than Coco's is

$$
7!\sum_{k=0}^{7} \frac{(-1)^{k}}{k!}=1,854
$$

(Why?) Thus, the probability is $1,854 / 7!=103 / 280$.
${ }^{1}$ The notation ${ }_{n} C_{k}$ stands for the number of ways to choose $k$ objects for a set of $n$. This is typically read " $n$ choose $k$." It is equal to $n!/(k!(n-k)!)$.

## Scissors Congruence

by Matthew Bates and Jane Wang

1. (a)

2. If we cut a polygon into finitely many pieces and rearrange the pieces, the resulting polygon has the same area (the sum of the areas of the pieces). Thus, two shapes with different areas cannot be scissors congruent.
3. Given any triangle, we can find an altitude of the triangle that is completely inside the triangle. Cut along the altitude as well as parallel to the base at half of the height of the altitude. Then rearrange the pieces to make a rectangle, as shown below. We can check that the resulting shape is indeed a rectangle by checking that the four angles are all right angles.

4. (a) Let us first label some lengths in our diagram and some congruent angles (which we know are congruent by considering how the diagonal passes through parallel sides).


The blue pieces are congruent because they are identical. We notice that the angles of the two red pieces are congruent, and so the two red triangles are similar. To show that they are congruent, it suffices to show that one of the sides is congruent. We will show that $y=h$.

To see this, we can notice by angle considerations that the orange triangle and the larger triangle including the orange triangle, the blue piece, and the white triangle are similar. Then, we have that $\frac{x}{\sqrt{h w}}=\frac{w-\sqrt{h w}}{w}$. Solving for $x$ gives us that $x=\sqrt{h w}-h$, and thus $y=h$ as desired. This shows that the red triangles in the two decompositions are congruent. Similarly, we can show that the orange triangles are congruent, giving us that this is a true scissors congruence.

We used that $h \leq w \leq 4 h$ when drawing the figure. Otherwise, the diagonal that we drew would not have broken our shapes into the pieces of the above figure (see below):

$w<h$

$w>4 h$
(b) We would like to show that every rectangle is scissors congruent to one that satisfies $1 \leq w / h \leq 4$. Suppose that we have a rectangle for which $w / h>4$. Then, we can cut it in half via a vertical cut and then stack the pieces on top of each other. This creates a new scissors congruent rectangle with $w^{\prime}=w / 2$ and $h^{\prime}=2 h$, giving us a ratio $\frac{w^{\prime}}{h^{\prime}}=\frac{w / 2}{2 h}=\frac{1}{4} \frac{w}{h}$. We can repeat this process to repeatedly divide the width/length ratio by 4 until it is between 1 and 4 inclusive. A similar process with horizontal cuts will work to convert a rectangle with $w / h<1$ to a scissors congruent rectangle with weight/height ratio between 1 and 4 .
(c) Using part (b), every rectangle is scissors congruent to one with $1 \leq w / h \leq 4$. Then, part (a) tells us that the can cut and paste the resulting rectangle into a square with equal area.
5. (a) We want to check that the pieces of the two squares indeed cut and paste to form a square. One way to see this is to put the shapes on a coordinate plane and confirm that the rearrangement gives the coordinates of a square.

(b) Starting with a right triangle with legs of length $a$ and $b$ and hypotenuse $c$, we can create the left shape. The left shape has area $a^{2}+b^{2}$ since it is made up of two squares of side lengths $a$ and $b$. The right shape has area $c^{2}$ since it is a square of side length $c$. Since the two shapes are scissors congruent and scissors congruent shapes have the same area, we have that $a^{2}+b^{2}=c^{2}$, giving a proof of the Pythagorean theorem.
6. (a) Every polygon is congruent to itself without needing any cut and paste operations.
(b) If we can cut and paste $P$ to $Q$, then reversing the process will give a scissors congruence of $Q$ to $P$.
(c) Overlay the decompositions of $Q$ coming from its scissors congruence to $P$ and its scissors congruence to $R$. This gives a sub-decomposition of $Q$ that can be rearranged to form either $P$ or $R$, giving us a way to rearrange $P$ to $Q$.
7. Here are possible triangulations of the shapes from the introduction.

8. We can first decompose the polygon into triangles and trapezoids as follows: through every vertex of the polygon, draw a vertical line. There may be multiple vertices on each line. Cutting along these lines gives us pieces that are trapezoids (if they have two vertical edges) or triangles (if they have one vertical edge). Any trapezoids can then be decomposed into triangles with a diagonal cut, giving us a triangulation of our polygon. An example of this process is shown below:

9. In this solution, we will use the word "triangulation" to refer to a triangulation whose vertices are all vertices of the original polygon. We induct on the number of sides of the polygon. If the polygon has 3 sides, then it is a triangle and is itself a triangulation.

Suppose that we have that any polygon with $\leq n$ sides has a triangulation. Now suppose that we have a polygon with $n+1$ sides. Consider the leftmost vertex and call it $p$ (if the leftmost vertex is not unique, first rotate the polygon slightly so that it is unique). Consider the triangle made up of $p$ and its two adjacent vertices on the polygon. If this triangle does not contain any other vertices of the polygon in its interior, then we can split the polygon into this triangle and another polygon with $n$ sides. The $n$-sided polygon can then by triangulated by our inductive hypothesis. Adding in the triangle containing $p$ gives a triangulation of our original polygon.

If the triangle containing $p$ contains another vertex of the polygon in its interior, then consider the left-most such vertex. We can draw a segment connecting $p$ and this vertex. Since our polygon was initially created with a single cycle of segments and therefore has no holes, cutting along this segment creates two polygons with $\leq n$ sides each. Each of these two polygons has a triangulation by our inductive hypothesis. Combining these triangulations gives us a triangulation of our original polygon.
10. Consider any polygon $P$. We first triangulate this polygon (Problem 8). Then, each triangle in this triangulation is then scissors congruent to a rectangle (Problem 3), and those rectangles are scissors congruent to squares of the same area (Problem 4). By iteratively applying exercise 5, the union of these squares is scissors congruent to a single square of the area of the original polygon. Using that scissors congruence is an equivalence relation (Problem 6), we have that $P$ is scissors congruence to a square of equal area. By the same process, any $Q$ of the same area is scissors congruent to the same square. Thus, we have that $P$ and $Q$ are scissors congruent since they are scissors congruent to the same polygon (Problem 6 again).

## Arts and Graphs

by Hanna Mularczyk

1. There are infinitely many valid answers. To check that the graph (hopefully with 7 vertices) is actually the one in the example, make sure their edge sets are identical. This problem might open an interesting discussion: What makes two drawings of the same graph different? If we're drawing by hand it's probably impossible to make sure we draw two vertices at the exact same location...
2. This one is very subjective (and there's really no limit to how unnice a graph can be).

Here are the drawings I came up with:

3. The graphs described in (a) and (d) are planar by the following two drawings, respectively:


The graphs described in (b) and (c) are nonplanar. These explanations can be informal. Perhaps the best way to go about them is to find some set of vertices that form a closed shape (like a triangle, square, etc.) and then argue that each of the other vertices must either be inside or outside of this shape, and then do casework. For example, in (b), vertices 1, 2, and 3 form a triangle, and moreover vertices 4 and 5 form an edge, and these two things can't cross, so either both vertices 4 and 5 must be inside the triangle or both must be outside the triangle. Continue this logic until we get a contradiction.
4. This table contains the requested data for each of the relevant graphs:

| Graph | Vertices ( $\boldsymbol{v}$ ) | Edges $(\boldsymbol{e}$ ) | Faces $(\boldsymbol{f})$ |
| :---: | :---: | :---: | :---: |
| My graph | 7 | 9 | 4 |
| 3(a) | 4 | 6 | 4 |
| 3(d) | 6 | 8 | 4 |

It may be difficult to see a relationship between the number of vertices, edges, and faces from just these 3 graphs, but if you make several more, you'll notice that the number of vertices, edges, and faces satisfy a nice formula! If you didn't find such a formula, we urge you to collect this data for more graphs and keep trying. (Also, skip the solution to Problem 5, because it contains a spoiler!)
5. Let $v, e$, and $f$ be the number of vertices, edges, and faces of the planar graph, respectively. We'll sketch a proof that $v-e+f=2$.

We will inductively remove edges.
If our planar graph drawing has no cycle (a path using edges of the graph that starts and ends in the same place), there is only one face. A cycle-less graph is known as a tree. A tree has one more vertex than it has edges, so the formula holds. (Why?)

If our planar graph drawing has a cycle, remove an edge from that cycle. This necessarily "merges" two faces into one, so the number of edges and faces both decreases by 1. Continue this process until we get a tree. The formula holds for the tree, and adding an edge and a face adds $\mathrm{a}-1$ and $\mathrm{a}+1$ on the left hand size of the formula, and thus does not change the formula, so it also holds for the initial graph before we removed edges.
6. One or both of the following:

7. One or both of the following:

8.


On the torus:

$$
\begin{aligned}
& v=5 \\
& e=10 \\
& f=4+1 \text { outer face }=5 \\
& \text { so, } v-e+f=5-10+5=0 . \text { No! }
\end{aligned}
$$

On the Möbius strip:

$$
\begin{aligned}
& v=6 \\
& e=9+1 \text { outer edge }=10 \\
& f=2+2 \text { outer faces }=4
\end{aligned}
$$

$$
\text { so, } v-e+f=6-10+4=0 . \text { No! }
$$

## Calendar

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Email:

Home Phone: $\qquad$ Cell Phone: $\qquad$
Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

The $\$ 50$ rate is for US postal addresses only. For international rates, contact us before applying.
Please check all that apply:Enclosed is a check for $\$ 50$ for a 1-year Girls' Angle Membership.I am making a tax-free donation.
Please make check payable to: Girls’ Angle. Mail to: Girls’ Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle @gmail.com.


A Math Club for Girls

## Girls’ Angle Club Enrollment

## Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls’ Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls’ Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome all girls (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.
How do I pay? The cost is $\$ 20 /$ meet for members and $\$ 30 /$ meet for nonmembers. Members get an additional $10 \%$ discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls’ Angle located? Girls’Angle is based in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to Girls’ Angle and send to Girls’ Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls’Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, founder and director of the Exploratory

Yaim Cooper, Institute for Advanced Study
Julia Elisenda Grigsby, professor of mathematics, Boston College
Kay Kirkpatrick, associate professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, assistant dean and director teaching \& learning, Stanford University
Lauren McGough, postdoctoral fellow, University of Chicago
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, associate professor, University of Utah School of Medicine
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Liz Simon, graduate student, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, associate professor, University of Washington
Karen Willcox, Director, Oden Institute for Computational Engineering and Sciences, UT Austin Lauren Williams, professor of mathematics, Harvard University

At Girls’ Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high-level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle any field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

## Girls’ Angle: Club Enrollment Form

Applicant's Name: (last) $\qquad$ (first) $\qquad$
Parents/Guardians: $\qquad$
Address: $\qquad$ Zip Code: $\qquad$
Home Phone: $\qquad$ Cell Phone: $\qquad$ Email: $\qquad$

Please fill out the information in this box.
Emergency contact name and number: $\qquad$

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? Yes No

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

Date: $\qquad$
(Parent/Guardian Signature)
Participant Signature:
Members: Please choose one.
$\square$ Enclosed is $\$ 216$ for one session (12 meets)I will pay on a per meet basis at $\$ 20 /$ meet.

> Nonmembers: Please choose one. $$
\quad \text { I will pay on a per meet basis at } \$ 30 / \text { meet. }
$$ $\square \quad$ I'm including $\$ 50$ to become a member, and I have selected an item from the left.

I am making a tax-free donation.

Please make check payable to: Girls' Angle. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

## Girls' Angle: A Math Club for Girls <br> Liability Waiver

I, the undersigned parent or guardian of the following minor(s)
do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.
$\qquad$ Date: $\qquad$ Print name of applicant/parent: $\qquad$
Print name(s) of child(ren) in program: $\qquad$


[^0]:    On the cover: Light Paths!, an ode to
    Katherine Knox's mathematical gem in the
    American Mathematical Monthly, by C.
    Kenneth Fan. See page 18.

[^1]:    ${ }^{1}$ This content is supported in part by a grant from MathWorks.

[^2]:    ${ }^{1}$ Katherine proved her theorem while in 7th grade. She is currently entering $10^{\text {th }}$ grade.

