Girls' Angle Bulletin

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To Foster and Nurture Girls' Interest in Mathematics

From the Founder

The creativity of Girls' Angle members is astonishing and inspiring! To this creativity, one adds a little bit of mathematical analysis, and mathematical discoveries ensue. The club is a veritable mathematics factory. - Ken Fan, President and Founder

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Jennifer Sidney
Executive Editor: C. Kenneth Fan

On the cover: *A well-ordering snake* by Juliette Majid. See page 19.

An Interview with Megan Kerr

Megan Kerr is the Katharine and Claudine Malone '63 Professor of Mathematics at Wellesley College. She is also an alumna of Wellesley College and earned her doctoral degree from the University of Pennsylvania under the supervision of Wolfgang Ziller.

This interview was conducted by Raegan Phillips and Ken Fan.

Girls' Angle: At what age did you discover your interest in mathematics? Was there someone who encouraged your curiosity in mathematics?

Megan: I always liked math, but it was only in college that I thought about focusing on math. Before then, if you asked me, I would simply have said that I liked school!

I discovered not long ago that back in first grade, in "My Book About Me" (a Dr. Seuss fill-in-the-blanks book), I wrote that Math was my favorite subject.

Girls' Angle: Could you please retrace your journey to becoming a professional mathematician? What were the most challenging difficulties that you had to overcome and how did you overcome them?

Megan: I went from college (at Wellesley College) straight to graduate school, at the University of Pennsylvania. I was very fortunate to have arrived at graduate school at a time when the math graduate program had a reasonable population of women. I made dear friends during those years and they supported me in a variety of ways, not only then, as we turned ourselves from

Many breakthroughs come when two mathematicians find a topic to which they each contribute some expertise, and combined, they have everything needed to solve the problem. This is a very nice way to learn new mathematics once you are out of graduate school.

students into mathematicians, but also more recently.

Graduate school was not easy! I did not sail through the "preliminary exam" (I did well the second time, but not well enough when I arrived). In fact, at the time, few entering students who came from US schools passed the first time, so that was fine. I also needed a second try to pass my "oral exam." However, I did have a supportive collection of faculty who encouraged me to persist, and I was working on my dissertation on schedule. I spent six years in graduate school; this had become the new normal in an era when jobs were scarce.

I got married in graduate school; my husband was a graduate student in Physics. We finished our doctoral studies in the same year. Since we were both applying primarily for postdoc positions, we did not try to land in the same place. It seemed more important to optimize during the (two year) postdoc period. It worked out for us, but he had to do a second postdoc, and there was some tension while he searched for a job near mine. As a Wellesley alumna, I was (I still am!) thrilled to be hired by Wellesley College. There are many schools in the Boston area, which was helpful for his job searching. But this area is also very desirable for researchers, which made it harder for him to stand out. This challenge, which we call 'solving the two-body problem' in academia, is so difficult for so many couples.

Dear Reader,

We're committed to producing quality math educational content and make every effort to provide this content to you for free.

We are also committed to surviving as a nonprofit!

For this issue, those who do not subscribe to the print version will be missing out on a portion of this interview with Prof. Megan Kerr and other content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

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Thank you and best wishes,
Ken Fan
President and Founder
Girls' Angle: A Math Club for Girls

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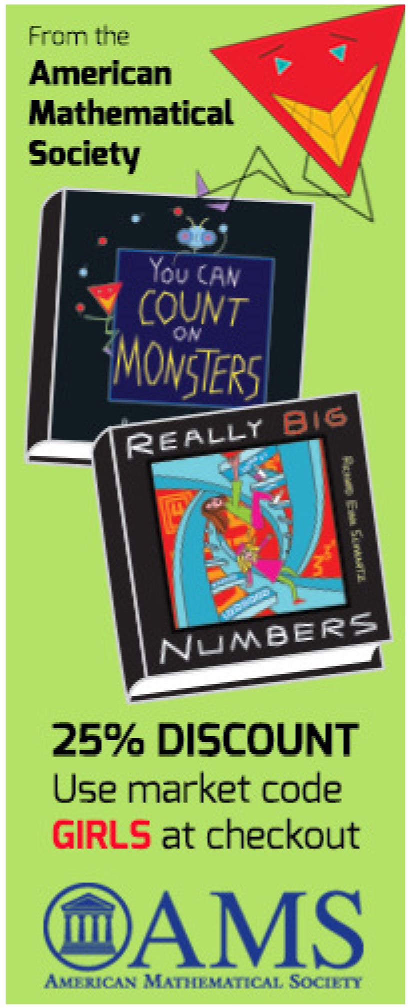

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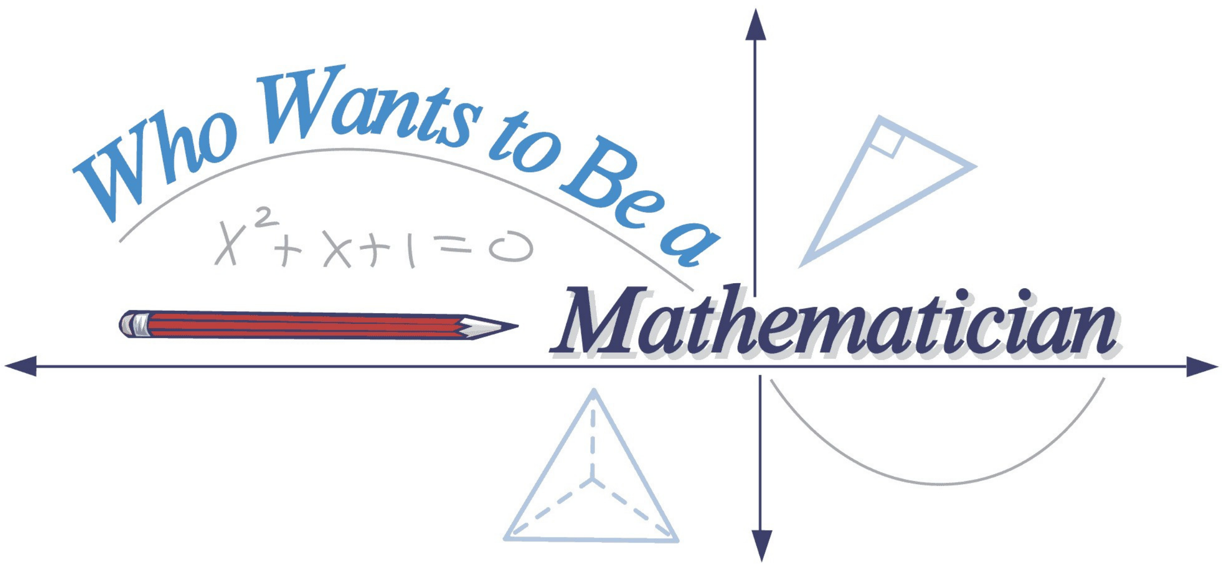
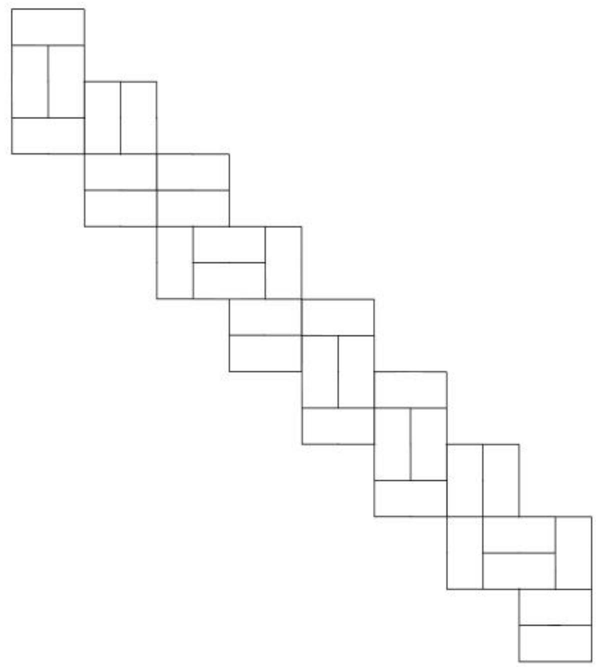


America's Greatest Math Game: Who Wants to Be a Mathematician.

Domino Tilings of a Zigzag

by Addie Summer | edited by Amanda Galtman

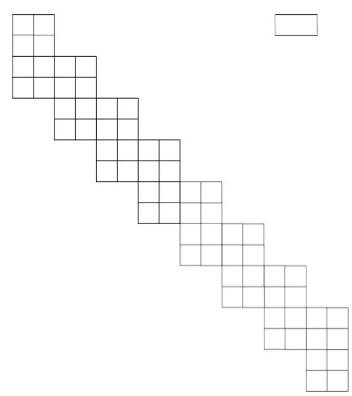
Nearby, there's a park with an old, tiled, zigzag path that looks schematically like this:



The actual path is in bad repair, with many tiles chipped and some even missing. City officials announced that they are going to dig out the old tiles and replace them, and they invited the public to submit a layout for a new domino tiling.

This invitation makes me wonder, just how many ways are there to tile this path with dominos?

To be crystal clear, how many ways are there to tile the shape below using 32 rectangular tiles each shaped like two squares joined along an edge?

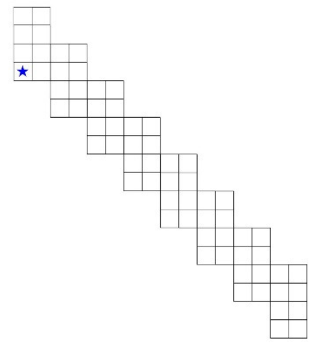


How many ways can this zigzag shape be tiled with dominos that are the shape of two squares joined together, as shown in the upper right?

To answer my question, the first thing I try to do is draw a few such tilings. It isn't long before I realize that there are many, many ways to tile that path – certainly, more than I want to draw by hand! Counting directly is out of the question.

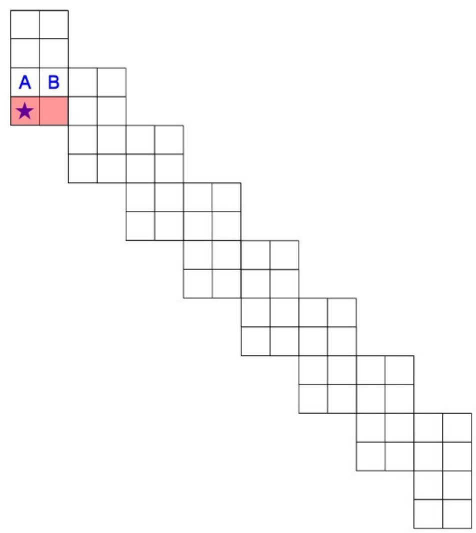
What can be done to find the answer?

Perhaps there's a way to organize the tilings into various types so that all the tilings of a certain type are easier to count. Then I can add up the numbers of tilings of each type to get the total. With this thought, my mathematical adventure begins.



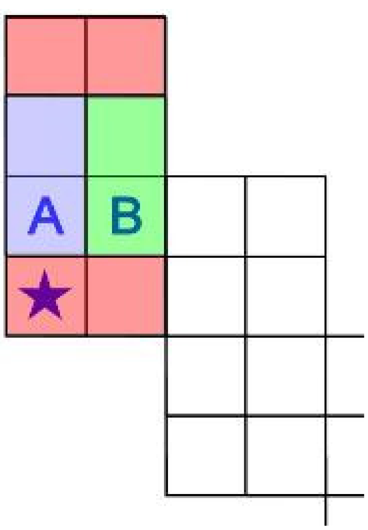
The square where the path first bends catches my attention. (The square is marked with a blue star at right.) And, in fact, that starred tile can be covered by a domino in only two ways. So, I could split the domino tilings into two types: the ones where the starred square is covered by a vertical domino and the ones where the starred square is covered by a horizontal domino. Every tiling must be of one of these two types, and no tiling can be of both. This organization seems promising, so let's think on it more. First, let's consider the tilings where the starred square is covered by a horizontal domino.

The horizontal types



If the starred square is covered with a horizontal domino, what else must be true?

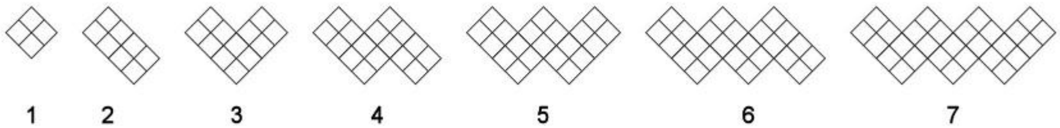
The square marked A can now be covered with a vertically oriented domino in only one way because such a tile cannot overlap with the tile covering the starred square below; it must be situated over the A and the square above it. And a horizontally oriented domino that covers the square marked A must be situated over the squares marked A and B, because there is no square to the left of A in the path. So there are only two ways to cover the square marked A.



If a domino is placed over squares A and B, the path splits into two separate pieces: a 2 by 2 square, and a path similar to the original, but shorter. That's progress, because it relates the tilings of the original path to tilings of shorter paths.

Let's see what happens if a vertical domino is placed over the square marked A. We're then forced to place a horizontal domino over the very top two squares and a vertical domino over the square marked B. We're again left with a shorter path akin to the original path. (See the figure at left, which focuses in on the top part of the path. The different colors are only there to clarify where the dominos are placed.)

In either case, we can relate domino tilings of the original path to domino tilings of shorter paths. This suggests that it would be helpful to consider all these shorter paths. Indeed, why not consider an entire family of zigzag paths that begin with the humble 2 by 2 square, where we attach 2 by 2 squares systematically as illustrated below?

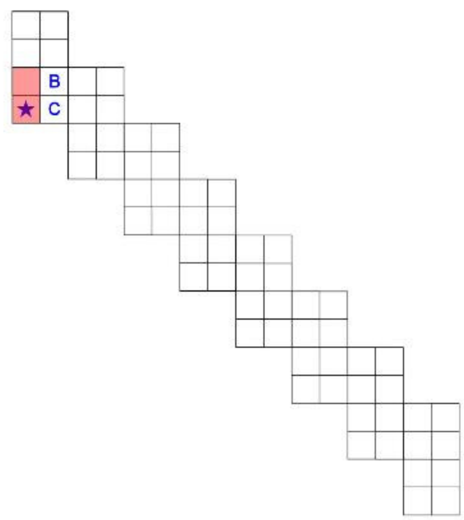


The path in the park corresponds to the 16th path in this sequence.

Let's denote by $N(k)$ the number of domino tilings of the kth such path. We want to know $N(16)$.

What we've seen so far is that the number of horizontal-type domino tilings of the 16th path is equal to $N(1)N(14) + N(14)$.

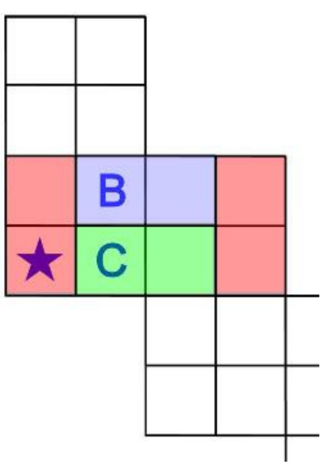
The vertical type



We still have to consider the case where we place a vertically oriented domino over the starred square, as shown in the figure at right. What can we say now?

We cannot place a vertical domino over the square marked B and the square above it. That would isolate a three-square section of the path, yet any path that is tiled with dominos must have an even number of squares. So, the square marked B can be covered in only two ways: a vertical domino over squares B and C or a horizontal square covering square B and the square to its right.

If a vertical domino is placed over squares B and C, the path splits into a 2 by 2 square and a shorter zigzag path. It's the same situation we found ourselves in when we put a horizontal domino over the starred square and another horizontal domino just above it.



If a horizontal domino is placed over square B, then we have no choice but to place a horizontal domino over square C, and a vertical domino to the right of those. (See the figure at left.) What remains uncovered consists of a 2 by 2 square and another, shorter, zigzag path.

So the number of vertical-type domino tilings of the 16th path is equal to $N(1)N(14) + N(1)N(13)$.

Adding up the paths of vertical and horizontal type, we find that $N(16) = N(1)N(14) + N(14) + N(1)N(14) + N(1)N(13)$.

Actually, we can perform this same analysis on all the zigzag paths, except the first three. For $n > 3$, the same analysis tells us that

$$N(n) = N(1)N(n-2) + N(n-2) + N(1)N(n-2) + N(1)N(n-3).$$

We can see directly that $N(1) = 2$, so substituting 2 for $N(1)$ in this equation gives us

$$\begin{aligned} N(n) &= N(1)N(n-2) + N(n-2) + N(1)N(n-2) + N(1)N(n-3) \\ &= 2N(n-2) + N(n-2) + 2N(n-2) + 2N(n-3) \qquad (*) \\ &= 5N(n-2) + 2N(n-3) \end{aligned}$$

For $n = 3$, the last term above would become $2N(0)$, but there is no path of length 0. To compute $N(3)$, let's revisit our analysis. This "path of length 0" arose when we were considering the tilings of vertical type at the bottom of the previous page, when we placed horizontal tiles over the squares marked B and C. We were then forced to put a vertical domino to the right of those, leaving us with a 2 by 2 path above and a path of length $n - 3$ below. If $n = 3$, there is no path below and we can complete the tiling to the entire path by using any tiling of the 2 by 2 path above. Instead of getting $N(1)N(n-3)$ domino tilings in this case, we get $N(1)$ tilings.

As a matter of convenience, we can simply declare $N(0)$ to be 1. Even though there is no meaning to this in terms of actual domino tilings and paths, this declaration makes our formula (*) work even when $n = 3$.

Knowing that $N(1) = 2$, if we can determine $N(2)$, we would then be able to use the formula (*) to compute as many values of $N(n)$ as we wish. The value of $N(2)$ is the number of domino tilings of the 2nd path in our family, which is a 2 by 4 rectangle.

I happen to know that the 2 by m rectangle can be tiled with dominos in exactly F_{m+1} ways, where F_k is the Fibonacci sequence which begins $F_1 = 1$, $F_2 = 1$, and satisfies $F_{k+1} = F_k + F_{k-1}$, for $k > 1$. So $N(2) = F_5 = 5$. If you're skeptical, by all means, please verify this directly!

The sequence

We can now compute $N(n)$ for several values of n by using (*):

n	0	1	2	3	4	5	6	7	8	9	10
$N(n)$	1	2	5	12	29	70	169	408	985	2378	5741

It's just a little more work to compute $N(16)$, which is the answer to our original question:

n	11	12	13	14	15	16
$N(n)$	13860	33461	80782	195025	470832	1136689

Wow! There are over a million ways to tile that little path in the park! I'm sure glad I didn't persist in trying to draw all of them! Even if I could draw one every minute, the task would take over two years to do.

A formula

Our formula (*) is an example of a linear recurrence relation, and we can use the theory of linear recurrence relations to obtain a formula for $N(n)$.

The idea is to first find geometric sequences that satisfy (*). Consider the geometric sequence with first term a and common ratio r:

$$a, ar, ar^2, ar^3, ar^4, \ldots$$

Which nonzero values of r result in a geometric sequence that satisfies (*) for any value of a?

To satisfy (*), we must have $ar^n = 5ar^{n-2} + 2ar^{n-3}$, for all $n > 2$. If we divide throughout by ar^{n-3}, we obtain the cubic equation $r^3 = 5r + 2$. So, any solution to this cubic equation gives us a value of r that can be used as the common ratio of a geometric sequence that satisfies (*).

The rational root theorem tells us that if the cubic $r^3 - 5r - 2$ has rational roots, they'll have the form p/q, where p is an integer that divides evenly into the constant term -2, and q is an integer that divides evenly into the leading coefficient 1. The only possibilities are 1, -1, 2, and -2. Let's plug each one into the cubic polynomial to see if it is a root:

$$\begin{aligned} (1)^3 - 5(1) - 2 &= -6 \\ (-1)^3 - 5(-1) - 2 &= 2 \\ (2)^3 - 5(2) - 2 &= -4 \\ (-2)^3 - 5(-2) - 2 &= 0 \end{aligned}$$

So, -2 is a root! We can factor out $r + 2$ from $r^3 - 5r - 2$ to obtain a quadratic polynomial, which we can then solve by using the quadratic formula to find the other roots.

The factoring

$$r^3 - 5r - 2 = (r + 2)(r^2 - 2r - 1)$$

enables us to find that the other two roots of the cubic are $1 \pm \sqrt{2}$.

Thus, for any a, b, and c, the geometric sequences $a(-2)^k$, $b(1+\sqrt{2})^k$, and $c(1-\sqrt{2})^k$ all satisfy the linear recurrence relation (*).

This means that $a(-2)^k + b(1+\sqrt{2})^k + c(1-\sqrt{2})^k$ also satisfies (*). (If two sequences satisfy the same linear recurrence relation, then the term-wise sum of the two sequences will yield a sequence that also satisfies the linear recurrence relation. If this fact isn't clear to you, please think about it and try to prove it. Also, see page 13 of Volume 3, Number 1 of the Girls' Angle Bulletin.) If we can find a, b, and c so that this sequence agrees with the sequence $N(k)$ for the first 3 values of k, then the two sequences must be identical, because each term is determined by the values of the previous 3 terms.

For this reason, we try to solve the system of linear equations

$$a(-2)^0 + b(1+\sqrt{2})^0 + c(1-\sqrt{2})^0 = 1 \qquad a + b + c = 1$$
$$a(-2)^1 + b(1+\sqrt{2})^1 + c(1-\sqrt{2})^1 = 2 \quad \text{which simplify to} \quad -2a + (1+\sqrt{2})b + (1-\sqrt{2})c = 2$$
$$a(-2)^2 + b(1+\sqrt{2})^2 + c(1-\sqrt{2})^2 = 5 \qquad 4a + (3+2\sqrt{2})b + (3-2\sqrt{2})c = 5$$

I'll spare you from my messy scratchwork and just show you the solution I found:

$$a = 0,\ b = \frac{2+\sqrt{2}}{4},\ \text{and } c = \frac{2-\sqrt{2}}{4}.$$

Do you agree?

Thus, $N(n) = \left(\frac{2+\sqrt{2}}{4}\right)(1+\sqrt{2})^n + \left(\frac{2-\sqrt{2}}{4}\right)(1-\sqrt{2})^n$.

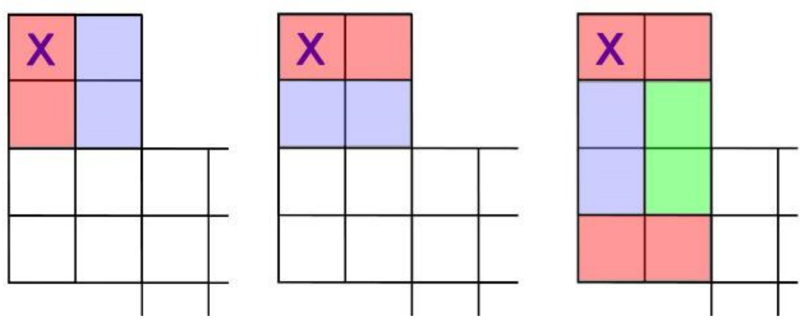
Wait a sec!

Our formula does not involve the root -2 of the cubic $r^3 - 5r - 2 = 0$ at all! (That is, $a = 0$.) It only involves the roots of its quadratic factor $r^2 - 2r - 1$.

That means that our sequence actually satisfies the linear recurrence $N(n) = 2N(n-1) + N(n-2)$.

We should be able to see this new recurrence in terms of tilings. That the recurrence formula only involves the previous two terms suggests building up the tiling earlier than the first bend (i.e., the square we earlier marked with a star). Let's go all the way to the top of the path.

The square marked "X" can be covered by a domino in only two ways, one horizontal and the other vertical. If we use a vertical domino, we have no choice but to place a vertical domino to its right, leaving a path shorter by one 2 by 2 square. If we use a horizontal domino to cover the X, we could place another horizontal domino just below, again leaving a path shorter by one 2 by 2 square. Another option is to place a vertical domino over the square just below the X, forcing us to put a vertical domino next to that and a horizontal domino below all three. This last option leaves a path two 2 by 2 squares shorter. That's it! Why didn't I just start at the top to begin with?



The three ways the top part of the path must be tiled that show why the linear recurrence relation $N(n) = 2N(n-1) + N(n-2)$ is true.

Isn't it neat how math finds ways to inform you of nifty things?

Compositions and Divisors

by Robert Donley[1]
edited by Amanda Galtman

In the previous installment, the cube model revealed some combinatorics of Boolean posets and gave some insight towards how mathematicians can grapple with higher dimensional objects. In this part, we extend these concepts directly to rectangular boxes and beyond. In turn, we obtain a way to count divisors of positive integers. It will be helpful for the reader to review the previous three installments, in particular the parts on basic properties of multiset counting, rectangular or block-shaped posets, and up linear operators.

Recall that the poset underlying Pascal's triangle consists of all ordered pairs (a, b) of nonnegative integers, and these points correspond to the integer points in the first quadrant of the xy-plane. If we fix some (a, b), all elements less than this element form a rectangular poset using the same partial ordering. To extend the Pascal's triangle concept, we consider ordered triples (a, b, c). Our first task is to describe how such triples give coordinates for points in three-dimensional space.

For our purposes, we assume x, y, and z are nonnegative real numbers; these (x, y, z) form the **first octant**, which we denote by $O(3)$. We also orient the directions of the axes as indicated in the left figure below. We'll plot the point (x, y, z) so that the first two coordinates are drawn in a horizontal plane, and the vertical height is determined by the z entry. See Figure 1 for examples.

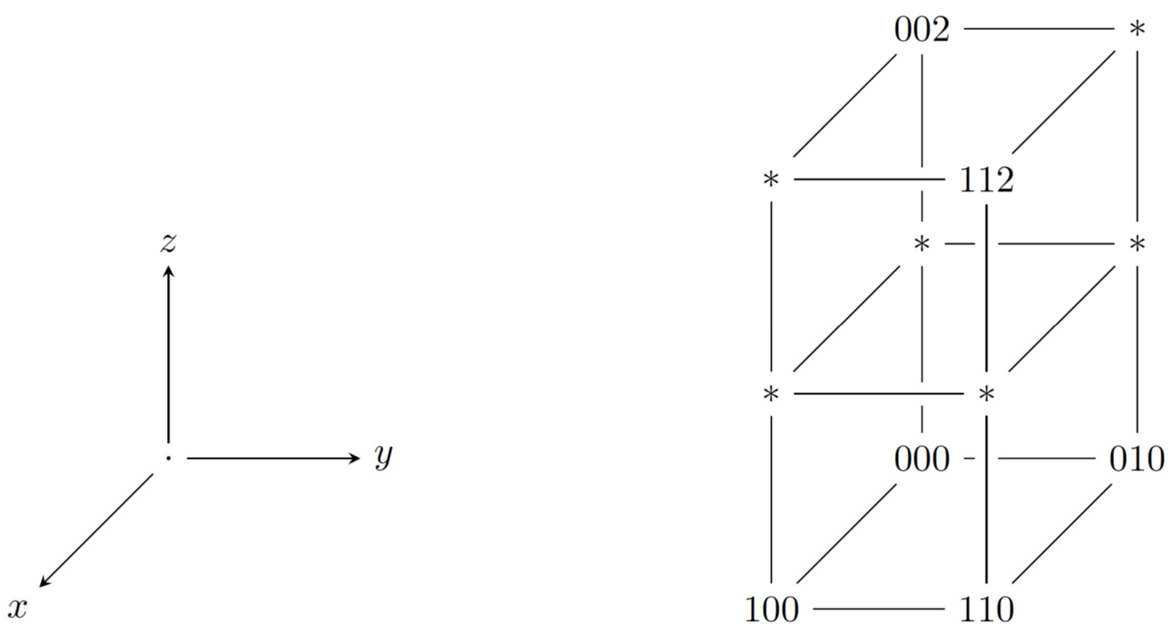


Figure 1. The first octant (left) and coordinates (a, b, c) written as abc (right).

[1] This content is supported in part by a grant from MathWorks.

Definition. Define $P(3)$ to be the poset of ordered triples (a, b, c) with nonnegative integer entries. For the partial ordering, we use entry-wise comparison; that is, $(a_1, b_1, c_1) \le (a_2, b_2, c_2)$ if and only if $a_1 \le a_2$, $b_1 \le b_2$, and $c_1 \le c_2$.

Features of $P(3)$ that extend Pascal's triangle are:

- $P(3)$ is the subset of integer points in the first octant, $O(3)$.
- The rank of (a, b, c) in $P(3)$ is $a + b + c$.
- Instead of horizontal line segments in a triangle, the levels of $P(3)$ are horizontal triangles in a pyramid, and
- The elements in $P(3)$ that cover an element q are given by increasing the value of one coordinate of q by 1; thus, each element in $P(3)$ is covered by three elements.

Figure 2 shows levels 0, 1, and 2, and their relative position in the first octant.

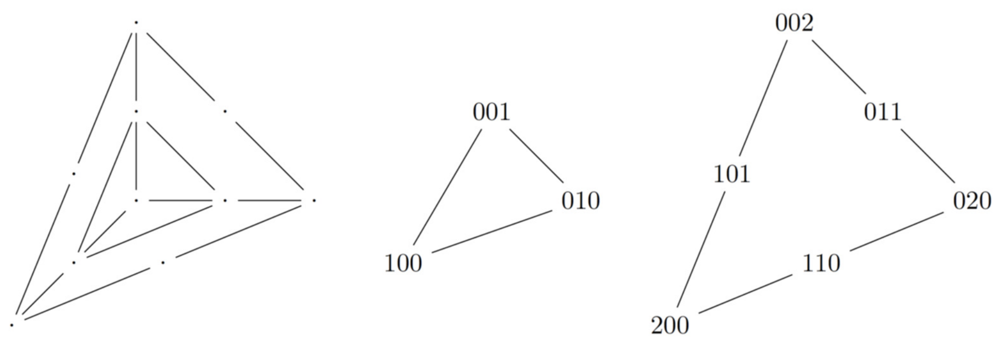


Figure 2.

Furthermore, chain counting yields a new notation that extends combinations. As we have seen in the previous two installments, a lattice path from (0, 0, 0) to (a, b, c) corresponds to a word in letters F (forward), R (rightward), and U (upward). From the matching theorem, there are

$$C(a,b,c) = \frac{(a+b+c)!}{a!b!c!}$$

such words. We use $C(a, b, c)$ instead of the traditional **multinomial coefficient** notation $\binom{a+b+c}{a,b,c}$ to simplify typesetting. When $c = 0$, the typical convention is to write the binomial coefficient as $\binom{a+b}{a}$.

Example: The number of maximal chains from (0, 0, 0) to (1, 1, 2) is given by $C(1, 1, 2) = 12$, represented by words

$FRUU$, $FURU$, $FUUR$, $RFUU$, $UFRU$, $UFUR$,
$RUFU$, $URFU$, $UUFR$, $RUUF$, $URUF$, $UURF$.

Exercise: Calculate $C(1, 1, 3)$ and list the corresponding words.

Exercise: Calculate $C(a, b, c)$ for each element in $P(3)$ with rank 0, 1, and 2. Explain how the symmetries of each level reduce the number of calculations.

Exercise: Draw the triangle for level 3 and list each triple in $P(3)$. Calculate each corresponding $C(a, b, c)$.

To obtain a finite graded poset in $P(3)$, we collect all elements less than a fixed maximal element (a, b, c). Instead of a rectangular grid, we now obtain the integer points of a rectangular box in $P(3)$.

Definition. We define the finite graded poset $R(a, b, c)$ to be the set of all (x_1, x_2, x_3) in $P(3)$ such that $(x_1, x_2, x_3) \leq (a, b, c)$. The partial ordering and grading on $R(a, b, c)$ are inherited from $P(3)$.

Note that the number of elements in $R(a, b, c)$ is $(a + 1)(b + 1)(c + 1)$.

Let's consider the Hasse diagram of the poset $R(1, 1, 2)$. In Figure 1 and in Figure 3 below, elements of $R(1, 1, 2)$ are the integer points of the rectangular box, and we tilt the box at the origin to emphasize the graded poset structure of $R(1, 1, 2)$. The levels correspond to elements on the same horizontal line in Figure 3.

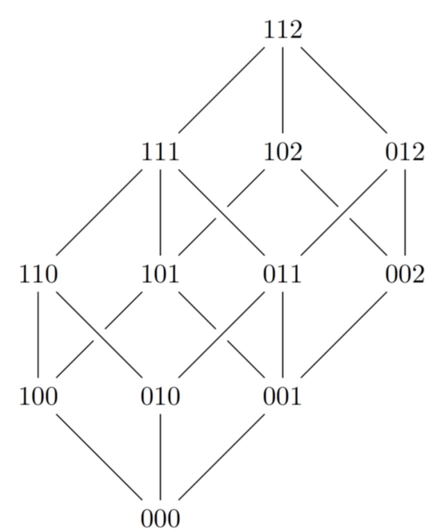


Figure 3.

Exercise: List all elements in $R(1, 2, 2)$, and verify the counting formula for the number of this poset's elements.

To continue with chain counting ideas, we note a version of Chu-Vandermonde convolution for rectangular boxes, which we exhibit for the special case of $R(1, 1, 2)$.

Exercise: For each level in $R(1, 1, 2)$, match each element with its dual element, which we define by $(x, y, z)^* = (1 - x, 1 - y, 2 - z)$. For instance, $(1, 0, 0)^* = (0, 1, 2)$.

Exercise: What happens if we apply the dual operation twice? How do the levels change under the dual operation? What happens to the Hasse diagram in Figure 3?

Fix a level with rank k. The key idea to the chain-counting proof of Chu-Vandermonde convolution is that each path from (0, 0, 0) to (1, 1, 2) intersects the level of rank k in exactly one point.

Exercise: Explain why the number of such paths through (x, y, z) is

$$C(x, y, z) \cdot C(1 - x, 1 - y, 2 - z).$$

Then verify that the sum of these products over the elements of rank k is $C(1, 1, 2)$. For instance, if $k = 1$, we get $C(1, 0, 0) \cdot C(0, 1, 2) + C(0, 1, 0) \cdot C(1, 0, 2) + C(0, 0, 1) \cdot C(1, 1, 1)$ which evaluates to $1 \cdot 3 + 1 \cdot 3 + 1 \cdot 6 = 12$.

Exercise: Repeat for $R(1, 2, 2)$ and $R(2, 2, 2)$.

While convenient in the case of Pascal's triangle, it quickly becomes cumbersome to record values of trinomial coefficients in three-space. Instead, we have the computational approach from the previous installment using linear algebra and the up linear operator. That approach also helps us generalize to a model with any number of coordinates.

For $P(3)$, define the up linear operator by linearly extending

$$U((x, y, z)) = (x + 1, y, z) + (x, y + 1, z) + (x, y, z + 1).$$

A similar formula holds for $R(a, b, c)$, although we omit any terms in the result not in $R(a, b, c)$.

Exercise: Calculate $C(a, b, c)$ for all elements in $P(3)$ at level $k = 4$ using the up operator and the values for $k = 3$ from above.

When all terms are defined, the analogue of Pascal's recurrence for $P(3)$ is

$$C(a, b, c) = C(a - 1, b, c) + C(a, b - 1, c) + C(a, b, c - 1).$$

In other words, $C(a, b, c)$ is obtained by adding the three entries adjacent to the position in the level below.

As seen in the previous installment, we recast the up operator as multiplication of polynomials to obtain

Trinomial formula: $(x + y + z)^k$ is the sum of the terms $C(a, b, c)x^a y^b z^c$ where (a, b, c) ranges over all triples such that $a + b + c = k$.

Exercise: Verify the values of $C(a, b, c)$ for levels $k = 1, 2, 3$ by computing $(x + y + z)^k$ directly.

Exercise: Set $x = y = z = 1$ in the trinomial formula. What formula results? Verify this formula directly for $k = 1, 2, 3$, and 4.

Exercise: Set $x = 1$, $y = 2$, and $z = -3$ in the trinomial formula. What formula results? Verify this formula directly for $k = 1, 2, 3$, and 4.

Exercise: Define the complex number $w = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. Show that $w^3 = 1$. Then verify that

$$w^3 - 1 = (w - 1)(w^2 + w + 1) = 0 \text{ and } w^2 + w + 1 = 0.$$

Set $x = 1$, $y = w$, and $z = w^2$ in the trinomial formula. What formulas result from the real and complex parts? Verify for $k = 1, 2$, and 3.

While our focus remains on triples, which are possible to visualize, the theory for general ordered tuples works mostly unchanged. We define some common terminology.

Definition: A **weak composition** of $k \geq 0$ into n **parts** is an ordered sum $x_1 + \ldots + x_n = k$ such that each $x_i \geq 0$. Similarly, for a (strict) **composition** of $k \geq n$ into n **parts**, we require each $x_i > 0$. For simplicity, we denote either type by the ordered n-tuple $(x_1, \ldots, x_n)$.

Example: A vertex of the k-cube is a weak composition of its rank where each x_i equals 0 or 1. The only vertex that is also a strict composition is $(1, \ldots, 1)$.

By the formula for multiset counting, there are $C(k + n - 1, k)$ weak compositions of k into n parts. In this case, we place k balls into n boxes.

Exercise: Verify the formula for the number of weak compositions in the cases where $n = 3$ and $k = 1, 2, 3$, and 4.

Exercise: For $k \geq n$, give a formula for the number of strict compositions of k into n parts.

Exercise: How many strict compositions are in $R(a, b, c)$?

Exercise: Where applicable, adapt the preceding model for $P(3)$ to $P(n)$, the poset of all compositions with n parts. Items of interest include rank and levels, multinomial coefficients for chain counting, the up linear operator, and the multinomial theorem. Repeat the exercises for $P(4)$ and $P(5)$.

Here's an example from number theory where compositions with many parts appear naturally.

Example: Fix a positive integer m. The elements of the poset D_m are the divisors of m, with the partial ordering defined by $x \leq y$ if x divides evenly into y. See Figure 4 for the Hasse diagrams of D_{72} and D_{30}. Do these Hasse diagrams look familiar?

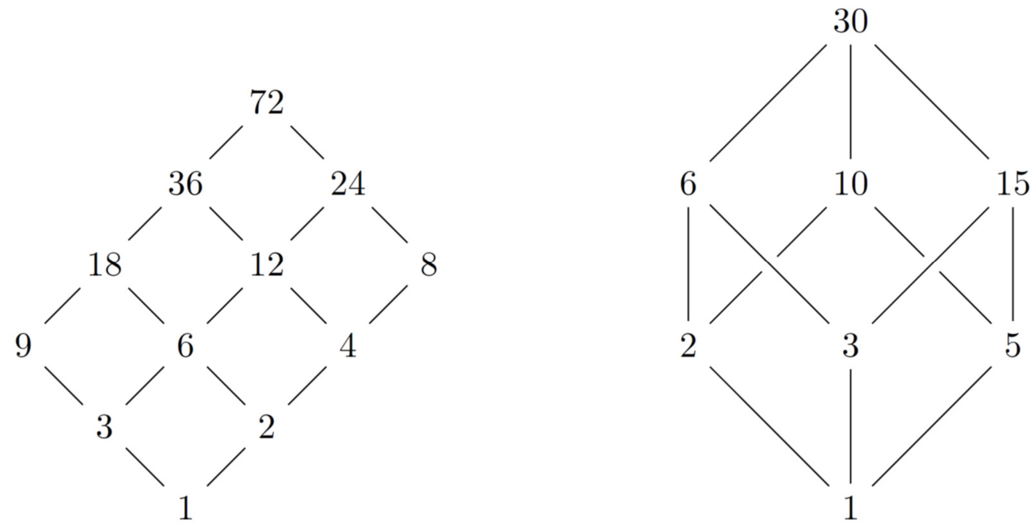


Figure 4.

The fundamental theorem of arithmetic states that every integer $m > 1$ factors uniquely into a product of powers of primes, up to order of the prime factors. Specifically, there exist primes $p_1 < \ldots < p_k$ and positive integers $e_1, \ldots, e_k$ such that $m = p_1^{e_1} \cdots p_k^{e_k}$. In turn, any divisor of m that is greater than 1 factors similarly using the same primes and with exponents f_i with $0 \le f_i \le e_i$. The divisors of m now correspond perfectly with weak compositions less than $(e_1, \ldots, e_k)$ by the identification

$$(f_1, \ldots, f_k) \leftrightarrow d = p_1^{f_1} \cdots p_k^{f_k}.$$

For example, in D_{72}, $p_1 = 2$ and $p_2 = 3$, so $24 = 2^3 3^1$ corresponds to the composition (3, 1).

Exercise: In D_m, explain what it means for y to cover x. If the rank of a divisor d is defined as the rank of its composition, give an interpretation of rank in terms of primes. Finally, what does a maximal chain in D_m represent in terms of primes?

Exercise: Convert the elements in Figure 4 to their corresponding compositions.

Exercise: Draw the Hasse diagram for the divisor poset of 150, and identify the composition for each element.

Exercise: In the notation of the fundamental theorem of arithmetic, how many divisors does the positive integer m have? Verify for m = 30, 72, 150, and 44,100.

Exercise: When do the Hasse diagrams for two divisor posets have the same shape?

Exercise: Describe the intersection of divisor posets $D_r \cap D_s$, where r and s are positive integers. What is the maximal element? How many divisors of s do not divide r?

Exercise: Let r and s be positive integers. What is the smallest positive integer t such that D_r and D_s are both contained in D_t?

Romping Through the Rationals, Part 3

by Ken Fan | edited by Jennifer Sidney

Emily and Jasmine are studying the sequence obtained by starting with 0, then successively applying the function $f(x) = 1/(1 + \lfloor x \rfloor - \{x\})$ to its terms. A social media post Jasmine read claims that every nonnegative rational number occurs in the sequence exactly once.

Emily and Jasmine are sitting at their favorite booth in Cake Country enjoying a thick, hearty slice of the most flavorful banana bread you could ever imagine.

Jasmine: How does Mr. Chemcake do it?

Emily: It's most definitely extra.

After the mind-numbing goodness wears off, they resume their math thread.

Jasmine: I feel like going back to the original sequence and trying to show once and for all that it really does produce all the nonnegative rational numbers.

Emily scrounges through her backpack and pulls out the sheet with the first few terms of the sequence they had computed earlier:

Recall that Emily and Jasmine had decided to start a new line after each integer that they encountered while computing the sequence.

0,
1,
1/2, 2,
1/3, 3/2, 2/3, 3,
1/4, 4/3, 3/5, 5/2, 2/5, 5/3, 3/4, 4,
1/5, 5/4, 4/7, 7/3, 3/8, 8/5, 5/7, 7/2, 2/7, 7/5, 5/8, 8/3, 3/7, 7/4, 4/5, 5,
1/6, 6/5, 5/9, 9/4, …

Emily: So far, we know for sure that every nonnegative rational number appears at least once in the sequence, and each fraction's denominator becomes the numerator of the next fraction. We also know that if $f(a) = b$ then $f(1/b) = 1/a$, unless a is an integer, in which case $f(a) = 1/(a + 1)$, but $f(1/(a + 1)) = (a + 1)/a$.

Jasmine: And we suspect that the sequence, after the initial zero, can be partitioned into subsequences that run from $1/n$ to n, where n is an integer. We know that this would follow if we could show that the nonnegative integers appear in the sequence in numerical order.

Emily: I think we need another observation to enable us to make additional progress.

Jasmine: I don't see anything else at the moment.

Emily: I forgot that we also conjectured that the integer n appears as the 2^n-th term in the sequence.

Jasmine: Oh, that's right! That's one of the first things we noticed. If that's true, that would imply that the subsequences from $1/n$ to n double in length each time you increase n by 1.

Emily: This doubling would be a remarkable property of the sequence. It must be a clue!

Jasmine: Hmm. What can we do with it?

Emily and Jasmine think. (Can *you* think of something?)

Emily: This sounds crazy, but maybe we can show that each term in the subsequence from $1/n$ to n somehow corresponds to two terms in the subsequence from $1/(n + 1)$ to $n + 1$?

Jasmine: Interesting – you're suggesting that the sequence can be organized like a binary tree. I'm doubtful since the sequence is generated by applying a function to the previous term to get the next term, which isn't very tree-like. But I don't have any other ideas, so we might as well try! Let's look at the subsequences for $n = 2$ and $n = 3$, because that would be the first occasion where it would not be clear which two terms in the subsequence from 1/3 to 3 would pair up with the two terms in the subsequence from 1/2 to 2 (which consists of just 1/2 and 2):

1/2, 2,
1/3, 3/2, 2/3, 3, …

Emily: Hey! I just tried the most straightforward thing – comparing the first two terms of the $n = 3$ sequence to the first term of the $n = 2$ sequence, and the last two terms of the $n = 3$ sequence to the last term of the $n = 2$ sequence – just to check if the terms in the sequence from $1/n$ to n might give rise to pairs of terms in the sequence from $1/(n + 1)$ to $n + 1$ in order. And look! The 1/2 gets split into 1/?, ?/2, and a 3 is placed where the question marks are. Similarly, the 2, which is 2/1, gets split into 2/?, ?/1, again with a 3 placed where the question marks are.

Jasmine: Wow! Can that really be true in general? If that pattern holds, the next eight terms should follow the pattern

1/?, ?/3,   3/?, ?/2,   2/?, ?/3,   3/?, ?/1

Emily: And they do! The next eight terms are 1/4, 4/3, 3/5, 5/2, 2/5, 5/3, 3/4, 4. So we replace the first two question marks with 4, the next two with 5, the next two with 5 again, and the last two with 4!

Jasmine: Amazing! Does it work for the next row, too?

Emily and Jasmine compare the row that begins with 1/4 to the next row, which begins with 1/5.

<table>
<tr><td colspan="2">1/4</td><td colspan="2">4/3</td><td colspan="2">3/5</td><td colspan="2">5/2</td><td colspan="2">2/5</td><td colspan="2">5/3</td><td colspan="2">3/4</td><td colspan="2">4/1</td></tr>
<tr><td>1/5</td><td>5/4</td><td>4/7</td><td>7/3</td><td>3/8</td><td>8/5</td><td>5/7</td><td>7/2</td><td>2/7</td><td>7/5</td><td>5/8</td><td>8/3</td><td>3/7</td><td>7/4</td><td>4/5</td><td>5/1</td></tr>
</table>

Emily: It does! I can't believe it!

Jasmine gets more excited.

Jasmine: In fact, it looks like the number that is inserted between the split numerator and denominator is their sum. For example, the 4/3 is split into 4/? and ?/3, and what number replaces the question mark? A 7, which is 4 + 3!

Emily: If we can prove this holds in general, that would yield a lot!

Jasmine: Yes, you might have found the key observation.

Emily: If this is true, then wherever the fraction P/Q occurs, where P and Q are relatively prime positive integers, we should later in the sequence find $P/(P + Q)$ followed by $(P + Q)/Q$. It shouldn't be too hard to show that $f(P/(P + Q)) = (P + Q)/Q$.

Jasmine: The number $P/(P + Q)$ is less than 1, so its floor is 0 and its fractional part is itself. That means $f(P/(P + Q)) = 1/(1 + 0 - P/(P + Q)) = (P + Q)/Q$. Great! That checks out nicely.

Emily: Let's try a proof by induction on n. Suppose we've managed to show, for all $1 \leq k \leq n$, that the sequence is built out of subsequences that go from $1/k$ to k, in numerical order of k, and that the subsequence from $1/k$ to k has 2^{k-1} terms with no integer appearing in between the $1/k$ and k. Furthermore, assume that up to this point, the pairing scheme we've noticed also works. Can we show that the next 2^n terms will go from $1/(n + 1)$ to $n + 1$ and also follow the pairing scheme?

Jasmine: I guess we have to show that the pairs spawned by each term in the subsequence from $1/n$ to n will appear consecutively. So we have to show that if P/Q, $f(P/Q)$ are two consecutive terms in the subsequence that goes from $1/n$ to n, then the last term of the pair corresponding to P/Q will yield the first term of the pair corresponding to $f(P/Q)$ when input into the function f.

Emily: We know that P/Q will spawn the pair of consecutive fractions $P/(P + Q)$, $(P + Q)/Q$, and we know that $f(P/Q)$ will also spawn a similar pair of consecutive fractions $M/(M + N)$, $(M + N)/N$, where $f(P/Q) = M/N$ in lowest terms. So, does $f((P + Q)/Q) = M/(M + N)$?

Jasmine: Let's see. I think we should express M and N in terms of P and Q.

Emily: Okay. We worked that out earlier. Hold on.

Emily searches through the stack of scratch paper in her backpack.

Emily: We found that $f(P/Q) = Q/(QK + Q - R)$, where K is the quotient and R is the remainder you get when you divide P by Q. So $M = Q$ and $N = QK + Q - R$. That means that M/N will correspond to the pair $Q/(QK + 2Q - R)$, followed by $(QK + 2Q - R)/(QK + Q - R)$.

Jasmine: We have to show that $f((P + Q)/Q) = Q/(QK + 2Q - R)$. So we need to determine the quotient and remainder obtained when we divide $P + Q$ by Q.

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What is Zero to the Zero?

by Ken Fan | edited by Jennifer Sidney

What is 0^0?

It's a common question, but the common answer, which is to explain that it is 1, misses the essence of mathematics. How and why? That's what we'll discuss in this article.

Why the Question, "What Is 0^0?"?

If we consider a positive base b, then at some point, we learn that $b^0 = 1$. If we set b to 0 in this equation, we get $0^0 = 1$.

On the other hand, if we consider a positive exponent n, then it is also true that $0^n = 0$, because if we multiply n zeroes together, we get 0. If we set n to 0 in this equation, we get $0^0 = 0$.

Different values for the same expression? That surely cannot be acceptable! Which one is correct? Hence, the question.

What Is 0^0?

And here's the answer:

> We are the creators of mathematics, and you can be, too.
> ***You** can define 0^0 to be whatever you wish!*

"Wait a sec!" you might be thinking. "Are you telling me that if I wanted to, I could define 0^0 to be 123?"

Yes! That *is* what I am saying!

Of course, very few people, if any, will adopt this definition. But, yes, you *can* define it so.

The only bounds to which mathematics is subject are the bounds of logic, and there is nothing illogical about declaring 0^0 to be 123. It is whimsical, but it is not illogical.

What Is Math?

You may find what I have just written difficult to accept, largely because of the way mathematics is taught in school. There, mathematics is viewed as a body of facts and procedures that have been handed down to us from previous generations. Students are taught those facts and procedures as if they were immutable truths – *the way things simply must be*.

But every mathematical fact and procedure, at some point in time, did not exist. Each was born only when somebody dreamt it up.

Some ideas turned out to be so beautiful or so useful that others adopted them. For example, the decimal number system for representing quantity has become nearly ubiquitous. But we don't have to go too far back in history to find completely different number representation systems being used by large civilizations of people.

And though we are taught that the symbol for the quantity 1 + 1 is written as "2," *you can, if you wish, invent your own way of representing numbers*. (And if you feel a compulsion to do so, go for it!)

In mathematics, you can invent new operations and new definitions. You can change old operations and old definitions. In mathematics, you are free to create math as you wish.

That's the spirit of math. That's what mathematicians are doing at this very moment: creating new math.

Of course, while you are free to create whatever math you wish, you cannot expect that people will adopt it. Naturally, it'll only be adopted if other people like it.

Mathematicians tend to like ideas that minimize arbitrary choices. Math that minimizes arbitrary choices is said to be **canonical**. Canonical math can leave the impression that the math is discovered and not created.

For example, if you did decide to define 0^0 to be 123, people are going to wonder, "Where on earth did the number 123 come from?" It doesn't seem to have any particular significance in the context of exponentials. It's an arbitrary choice, so it is **noncanonical**.

Canonicality in math is not the only criterion that affects how mathematicians will receive an idea. Examples of other criteria are elegance, beauty, and significance.

The Most Common Way Teachers Respond to the Question

Most teachers say that 0^0 is 1. They draw your attention to the function $f(x) = x^x$, and they prove that the limit, as x tends to 0, of $f(x)$ is 1.

But this does *not* prove that $0^0 = 1$! There's nothing particularly special about the function $f(x)$ for it to be used as the arbiter on how 0^0 should be defined. We could construct another function of the form $g(x)^{h(x)}$, where both $g(x)$ and $h(x)$ tend to 0 as x tends to 0, but where the limit of $g(x)^{h(x)}$ is different from 1 – including, for instance, having a limit of 123!

To be clear: it does not follow logically that $0^0 = 1$ from the fact that $\lim_{x\to 0^+} x^x = 1$. Indeed, when we compute the limit of a function $f(x)$ as x tends to 0, the value of f at 0 is irrelevant; it need not even exist.

(It's worth noting that nobody argues that 0/0 should be 1 because $\lim_{x\to 0^+} x/x = 1$.)

So, why do people tend to declare 0^0 to be 1? In truth, it's just a matter of convenience. For example, there are formulas that would work nicely if 0^0 is declared to be 1.

Zero Factorial

Asking for the value of 0! is an entirely analogous question. Factorial is introduced as a function on positive integers n: n factorial is defined to be the product of the integers from 1 to n, inclusive.

It isn't long, however, before we start to write down formulas for which giving meaning to 0! becomes convenient. For example, the coefficient of $x^k y^{n-k}$ in the expansion of $(x + y)^n$ is given by the formula $\frac{n!}{k!(n-k)!}$ for $0 < k < n$. In this context, it makes sense to talk about the coefficient of $x^0 y^n$ in that expansion. But if you set k to 0 in the formula, you get 0! in the denominator. As a matter of convenience, we can declare that $0! = 1$ so that the formula works even when $k = 0$ (or $k = n$).

There's also the gamma function, which is a function defined for almost all complex numbers that returns $n!$ when you plug in $n + 1$ as input. The gamma function returns 1 when you input 1.

Still, anything can happen in the future! Perhaps some mathematician will discover an important but different function that extends factorial, yet suggests a different value for 0!. Maybe you'll dream up a scenario where it becomes convenient for you to declare that 0^0 be 0 instead of 1. Should that transpire, don't forget that you're perfectly free to do so! All you have to do is say early on in your paper that you find it convenient to define 0^0 to be 0. The quality of the mathematical ideas in your paper will provide the justification.

Summary

Once we define, for positive integers b and n, that $b^n = b \times b \times b \times \ldots \times b$, where there are n factors of b in the product, it follows by force of logic that $b^n b^m = b^{n+m}$ for positive integers b, n, and m. So the exponential rule $b^n b^m = b^{n+m}$, for positive integers b, n, and m, is not something we can revise, given our definition of the exponential b^n.

However, when we wish to extend exponents to all integers for the first time, we *are* free to define them however we please.

A long time ago, someone had the fruitful idea of trying to extend exponents to all integers in a way that preserves the rule $b^n b^m = b^{n+m}$. This person discovered that it is possible to do that, and, for positive integers b, doing so forces $b^0 = 1$ and $b^{-n} = 1/b^n$. This way of extending exponents to all integers has shown itself to be quite useful, so it has been adopted and is taught in schools as if it were an unalterable fact.

But you really *can* alter it. Mathematics is not a fossilized collection of facts. It is a creative art, not a science. Mathematics undergoes constant revision and renewal, and *you* can be a creator and a reviser of math.

We are the creators of mathematics, and *you* can be, too!

Notes from the Club

These notes cover some of what happened at Girls' Angle meets. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

Session 32 - Meet 1 January 26, 2023	Mentors:	Elisabeth Bullock, Cecilia Esterman, Anne Gvozdjak, Abhilasha Jain, Hanna Mularczyk, AnaMaria Perez, Laura Pierson, Vievie Romanelli, Jane Wang, Jing Wang, Rebecca Whitman, Muskan Yadav, Angelina Zhang

Take an origami square and roll it up so that two diagonally opposite corners touch. Arrange it so that the paper forms part of a cylinder. Hold it so that the touching corners are located at the highest point and the diagonal connecting the other two corners are the lowest points. Now project the paper edge onto a vertical wall using lines that are horizontal and perpendicular to the axis of the cylinder. Can you show that the projected curve is a graph of the cosine function?

Session 32 - Meet 2 February 2, 2023	Mentors:	Elisabeth Bullock, Jade Buckwalter, Anne Gvozdjak, Abhilasha Jain, Hanna Mularczyk, AnaMaria Perez, Laura Pierson, Vievie Romanelli, Jane Wang, Rebecca Whitman, Muskan Yadav, Angelina Zhang

Self-referential tests are making a comeback at the club! For an example of one, check out pages 20-21 of Volume 11, Number 2 of the Girls' Angle Bulletin. There, you'll find a 15-question multiple choice self-referential test created by four Girls' Angle members.

Session 32 - Meet 3 February 9, 2023	Mentors:	Elisabeth Bullock, Jade Buckwalter, Anne Gvozdjak, Clarise Han, Hanna Mularczyk, Kate Pearce, AnaMaria Perez, Laura Pierson, Jane Wang, Jing Wang, Rebecca Whitman, Muskan Yadav

The following question is based on a beautiful question dreamed up by member Eleanor. You create a random triangle by picking three numbers a, b, and Z between 0 and 1, uniformly and independently at random. The triangle has two sides of length a and b and the angle included between these two sides measures πZ radians. What is the probability that the resulting triangle is obtuse?

Session 33 - Meet 4 February 16, 2023	Mentors:	Elisabeth Bullock, Jade Buckwalter, Anne Gvozdjak, Clarise Han, Hanna Mularczyk, Kate Pearce, AnaMaria Perez, Laura Pierson, Vievie Romanelli, Rebecca Whitman, Muskan Yadav

If the coordinates of the points are (x_1, y_1) and (x_2, y_2), then the Euclidean distance between them is given by $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. So many facts can be deduced from the Euclidean distance formula between two points in the plane. For just a few, can you deduce the angle sum formulas for cosine and sine? Can you find the area of a parallelogram with vertices at (0, 0), (a, b), (c, d), $(a + c, b + d)$? Can you prove the triangle inequality? How many results can you prove by applying this formula?

Calendar

Session 31: (all dates in 2022)

September	8	Start of the thirty-first session!
	15	
	22	
	29	
October	6	
	13	
	20	
	27	
November	3	
	10	
	17	
	24	Thanksgiving - No meet
December	1	

Session 32 (tentative schedule): (all dates in 2023)

January	26	Start of the thirty-second session!
February	2	
	9	
	16	
	23	No meet
March	2	
	9	
	16	
	23	
	30	No meet
April	6	
	13	
	20	No meet
	27	
May	4	

Girls' Angle has run over 150 Math Collaborations. Math Collaborations are fun, fully collaborative, math events that can be adapted to a variety of group sizes and skill levels. We now have versions where all can participate remotely. We have now run four such "all-virtual" Math Collaboration. If interested, contact us at girlsangle@gmail.com. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

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Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ____________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

The $50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

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- ☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

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Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is $20/meet for members and $30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle is based in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: ________________________

(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is $216 for one session (12 meets) ☐ I will pay on a per meet basis at $20/meet.	☐ I will pay on a per meet basis at $30/meet. ☐ I'm including $50 to become a member, and I have selected an item from the left.

☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: ________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____________________________________