Girls' Angle Bulletin

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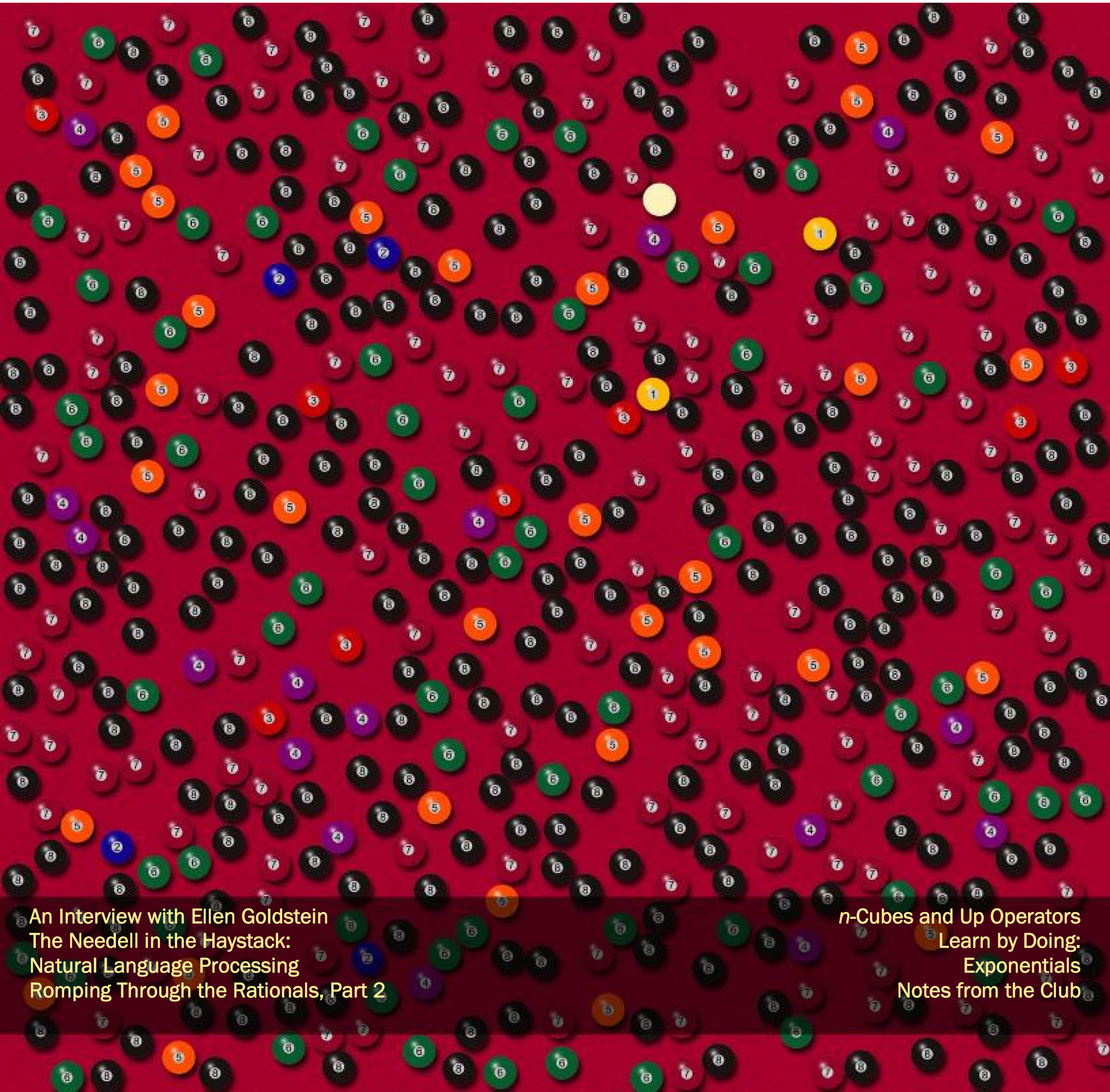
To Foster and Nurture Girls' Interest in Mathematics



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From the Founder

At Girls' Angle, mentor's role has evolved. Today, a critical function they play is to capture thoughts. At any given meet, many would drift off into oblivion, if it weren't for our mentors. By catching them, they save dreams and build member's self-belief. - Ken Fan, President and Founder

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Jennifer Sidney
Executive Editor: C. Kenneth Fan

Girls' Angle: A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: *Exponential Pool* by C. Kenneth Fan. There are 2^n balls labeled n plus a cue ball. What percentage are 8 balls? For more on exponentials, see page 23.

An Interview with Ellen Goldstein

Ellen Goldstein is Assistant Professor of the Practice of Mathematics at Boston College. She received her Doctor of Philosophy in Mathematics from Tufts University under the supervision of George McNinch. She graduated from Skidmore College with a double major in mathematics and dance. She dances with Dance Prism, a nonprofit ballet company.

This interview was conducted by Girls' Angle's Raegan Phillips and Ken Fan.

Girls' Angle: When did you start to notice mathematics? Can you recall one of the first mathematical things that caught your attention?

Ellen: I remember learning how to expand binomials and recognizing the pattern in the exponents and leading coefficient. Pascal's Triangle is still one of my favorite topics, particularly because it can be used in so many different settings.

Girls' Angle: Could you please describe your own personal journey into professional mathematics?

Ellen: I think it began in 9th grade when my algebra teacher wrote "I want you in math league" at the top of one of my assignments. I enjoyed working on the challenging problems after school and traveling to different schools to compete. In college, I took an introductory number theory course and thought that the material was absolutely beautiful. At the same time, I applied for and won a national scholarship for women in mathematics (the Clare Boothe Luce scholarship). Both of these things really catalyzed my intention of majoring in

My main goal for my students is that they leave the class with the confidence and skills they need to learn more math in the future, either in another class or on their own.

mathematics, rather than some of my other interests. When I was thinking about what to do after college, I wanted to keep learning math and remember being amazed that programs would pay you to get a PhD. Once again, it was this combination of not only loving the math and also seeing it as a viable career path that fermented my choices. I had tutored some during college but in graduate school, I started teaching my own classes and treasured the connections I made with students and how much I was able to support their learning. I had excellent professors in college who set the model for the kind of teacher I wanted to be. I applied for lots (LOTS) of jobs at the end of graduate school and got a teaching postdoctoral position at Northwestern University. While there, I participated in Project NExT, a professional development program through the Mathematical Association of America for new or recent PhDs in the mathematical sciences. This gave me a vibrant network of similarly-minded mathematicians who have supported me and inspired me while I found my place within the profession. I'm currently an Associate Professor of the Practice of Mathematics at Boston College, where my duties center around undergraduate instruction and helping graduate students learn how to teach.

Girls' Angle: What's your favorite thing about being a mathematician?

Ellen: I love patterns and puzzles. Being a mathematician for me means sharing math with students and seeing it through their eyes. I'm constantly getting new

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We're committed to producing quality math educational content and make every effort to provide this content to you for free.

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For this issue, those who do not subscribe to the print version will be missing out on a portion of this interview with Prof. Ellen Goldstein and other content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

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Thank you and best wishes,
Ken Fan
President and Founder
Girls' Angle: A Math Club for Girls

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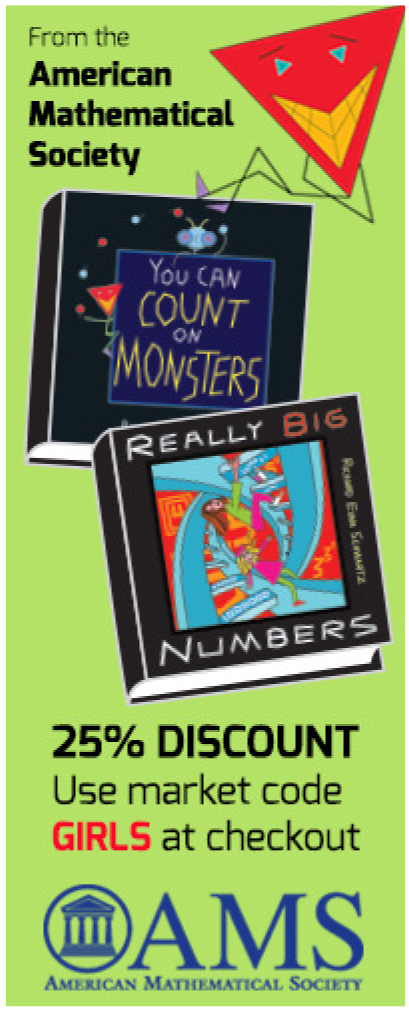

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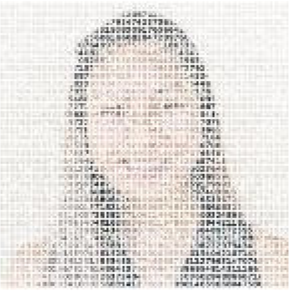
America's Greatest Math Game: Who Wants to Be a Mathematician.



The Needell in the Haystack[1]

Natural Language Processing:
A Princess, a Dragon, and a Simple Model
by Anna Ma | edited by Jennifer Sidney

In a far-off kingdom, there lived a beautiful princess named Elizabeth who was renowned for her intelligence and wit. She spent her days studying the many different languages and cultures of the world, and she dreamed of using her knowledge to bring peace and understanding to her kingdom.

One day, Elizabeth learned about a new field called natural language processing, or NLP. She was fascinated by the idea of using computational techniques to analyze and understand human language, and she knew that this was the field she wanted to pursue. Elizabeth threw herself into her studies, determined to become an expert in NLP. She spent long hours reading and practicing, and she quickly became one of the top students in her class.

As she progressed in her studies, Elizabeth began to develop a reputation as a brilliant and innovative thinker. She was always coming up with new ideas and approaches to solving language-related problems, and her professors and classmates were constantly impressed by her intelligence and creativity.

But as Elizabeth worked to bring peace and understanding to her kingdom, she faced many challenges. One of the biggest challenges came in the form of a fierce dragon that had long terrorized the land. Elizabeth knew that she had to find a way to stop the dragon and bring peace to her kingdom. She decided to use her skills in NLP to communicate with the dragon and try to understand its motivations.

Through careful study and analysis, Elizabeth was able to determine that the dragon was actually just misunderstood, and that it was acting out due to a misunderstanding of human behavior. With this knowledge, Elizabeth was able to use her NLP skills to broker a peace agreement between the dragon and the humans, and bring an end to the conflict that had long plagued her kingdom.

Figure 1. Princess Elizabeth and her dragon friend.

Elizabeth became known as the "Language Princess," and her fame spread far and wide. She was hailed as a hero and a visionary, and her work had a profound impact on the kingdom, helping to create a more united and understanding society. And with the dragon now at peace, Elizabeth lived happily ever after, using her skills and knowledge to bring people closer together through the power of language and communication.

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[1] This content is supported in part by a grant from MathWorks. Anna Ma is a Visiting Assistant Professor at the University of California Irvine.

A Story without an Author and an Image without an Artist

You might be surprised to learn that the above story and image have no (human) author/artist; they were both *artificially generated* by machine learning tools. The story was generated by a program called chatGPT, a chatbot NLP (Natural Language Processing) model developed by OpenAI.[2] To prompt chatGPT for the story, all I had to do was type into a text prompt – "write a story about NLP with a princess and a dragon" – and the story was miraculously generated! I was able to read the story while chatGPT generated the text, and I was excited to watch it evolve. In the first half of the story, I was worried that chatGPT would not recognize that I wanted both a princess and a dragon. To my delight, however, it did fit both into the story. The image was generated by a model developed by DeepAI[3], obtained by typing "princess with dragon and book" into a text prompt and choosing "fantasy world style" as my desired art style. You can try both links and see what kinds of stories, articles, and pictures these technologies can develop from just a simple prompt!

But how do these applications work? Is there a human on the back end speedwriting stories on demand and generating digital images of princesses with dragons? Perhaps at one point, this was the only way to accomplish such a feat. But with advances in computation, mathematics, and machine learning, companies have successfully implemented algorithms that can complete these tasks without human supervision. In the next few installments of "Needell in the Haystack," we will dive into natural language processing, image processing, and neural networks to better understand the algorithms behind these applications. In this edition, we will focus on an introduction to natural language processing and simple language models.

Natural Language Processing

Natural Language Processing is at the cross-section of Linguistics (the study of language and its structure) and Computer Science. NLP is also a subfield of machine learning that focuses on teaching computers to understand, interpret, and generate human language.

You've probably already seen NLP in action. In Microsoft Word or Google Docs, autocorrect suggests corrections to your essays well beyond misspellings: remedies for grammatical errors, wordiness, and much more are proposed in these word processing editors. If you have a smart home device such as an Amazon Alexa or Google Home, the device must process the audio of human language, interpret it, and then generate a response in a meaningful way. We will further explore the auto-complete functions on your phone and email that make texting/writing emails easier.

To perform these tasks, NLP models typically rely on a combination of linguistics, computer science, and machine learning techniques. They might use techniques such as tokenization (splitting a sentence into individual words), stemming (reducing a word to its root form), and part-of-speech tagging (identifying the role a word plays in a sentence) to analyze and understand the structure and meaning of language data.

We first focus on a simple auto-complete model to demonstrate the interplay between words/text and **transition matrices**. The idea here is to generate the next word based on the likelihood of

[2] https://chat.openai.com/chat
[3] https://deepai.org/machine-learning-model/fantasy-world-generator

the next word occurring. What follows is a simple example:

Fill in the sentence with one of the following words {mango, fun, squishy}:

I think math is very _________.

We might argue that only the word "fun" makes sense in this context. This inference is based on prior exposure to reading, writing, speaking, and listening to others converse, as well as our understanding of the meaning of the words. What if we are instead given the following words to choose from: challenging, book, rewarding? In this case, one can argue that two of the possible options can reasonably complete the above sentence, but whether you would complete it with "rewarding" or "challenging" might depend on your point of view of mathematics. To develop a method that will auto-complete a sentence, we first look for a way to assign probabilities to the likelihood of words appearing in a sentence.

A Simple Language Model

As in any machine learning technique, we start with data. In NLP, our data typically comes in the form of text, so we consider the following toy data set:

"I think math is very fun. I think math is fun. My mango is squishy. Math has challenging and fun problems. It is challenging if my mango is not squishy. I find math very fun and rewarding. I have fun when I have my squishy mango."[4]

To process this data set, we first must choose how to break up the text into individual components. This is known as **tokenization**. A natural way to tokenize the English language is by splitting text into individual words, which we refer to as **tokens**. Another approach for tokenization breaks text into sequences of letters of length n for some n. The choice of tokenization can affect the model. For example, the words "artistic" and "artisan" are two different tokens when tokens represent words; but if tokens are taken to be 4-letter sequences, these two words would be regarded as the same since both words begin with "arti." Because we want to design a model that will auto-complete words in a sentence (and also for simplicity), we tokenize by words and thus consider the tokens

$$T = \{\text{think, math, fun, mango, squishy, challenging, problem, rewarding}\}$$

Note here that we exclude words such as "I," "is," and "my." In NLP, extremely common words are sometimes removed before processing. When there is such a list of words that are filtered out, they are known as **stop words**. Although not all NLP tools remove stop words completely (or at all), we remove them here for simplicity and demonstrative purposes. We also ignore capitalization.

Consider a $|T| \times |T|$ matrix where each row and each column represent a word in our set of tokens T. We denote this matrix by M and interpret it as follows: M_{ij} will represent the probability that word j follows after word i. How can we use our data to compute these probabilities? To do so, let's start with a clean slate by initializing our matrix with all zeros, as shown here:

---

[4] Anna recently was gifted a squishy mango-shaped stress ball, and she loves it – as you can see!

	think	math	fun	mango	squishy	challenging	problem	rewarding
think	0	0	0	0	0	0	0	0
math	0	0	0	0	0	0	0	0
fun	0	0	0	0	0	0	0	0
mango	0	0	0	0	0	0	0	0
squishy	0	0	0	0	0	0	0	0
challenging	0	0	0	0	0	0	0	0
problem	0	0	0	0	0	0	0	0
rewarding	0	0	0	0	0	0	0	0

There are many different ways to fill in this matrix. One way is to count the number of times word *i* is followed by word *j* and divide that by the total number of times word *i* appears before any word. Doing so gives us an empirical likelihood that two words appear together in a sentence. Consider our data set after stop words have been removed:

> "think math fun, think math fun, mango squishy, math challenging fun problems, challenging mango squishy, math fun rewarding, fun squishy mango."

Then our co-occurrences (word pairs) are

> think math // math fun // think math // math fun // mango squishy // math challenging // challenging fun // fun problems // challenging mango // mango squishy // math fun // fun rewarding // fun squishy // squishy mango.

There are a total of four co-occurrences starting with the word "math," and three of them are followed by "fun." Thus $M_{\text{math, fun}} = 3/4$. As another example, the word "mango" appears first in two co-occurrences and is followed by the word "squishy" in both of them, so $M_{\text{mango, squishy}} = 2/2$. As in the last example, the word "fun" appears first in three of the co-occurrences and is followed by the word "rewarding" once, so $M_{\text{fun, rewarding}} = 1/3$. The matrix below represents our data set as a **transition matrix**, which is a matrix containing the probability of transitioning from one word to the next.

	think	math	fun	mango	squishy	challenging	problem	rewarding
think	0	1/1	0	0	0	0	0	0
math	0	0	3/4	0	0	1/4	0	0
fun	0	0	0	0	1/3	0	1/3	1/3
mango	0	0	0	0	2/2	0	0	0
squishy	0	0	0	1/1	0	0	0	0
challenging	0	0	1/2	1/2	0	0	0	0
problem	0	0	0	0	0	0	0	0
rewarding	0	0	0	0	0	0	0	0

We are ready to auto-complete some sentences with this very simple language model! Let's start with our first example:

I think math is very _________.

After removing the stop words, we find we are looking for a word to follow "math." Looking at our transition matrix, the most likely word is "fun" since $M_{\text{math, fun}} = 3/4$, thus our autocompleted sentence is

I think math is very **fun**.

Here are some others you can try on your own:

1. Solving problems makes math _________.
2. I love when I have my squishy _________.
3. I love challenging math _________.
4. I think all _________ should be squishy!
5. I think problems that are challenging are _________.
6. The number of even numbers from 1 to 10 is _________.

You may have noticed that some answers sound wrong or impossible to answer. Our auto-complete for 4 in particular, "I think all **math** should be squishy," does not make sense. There is some ambiguity in our auto-complete for 5 because "fun" and "mango" are equally likely to be the next word. These particular issues arise because our data set is so small; we have very limited information about completing sentences (especially sentence 6!). With more data, probabilistic ties would also become less likely, unless the words were used together precisely the same number of times (in which case, they may be interchangeable). Another issue that may arise with a simple model is that there is no way to address any necessary grammatical changes to the text. For example, "mango" would need to be changed to "mangos" in order to complete 4. Furthermore, this transition matrix assumes that the next word only depends on the previous, so it cannot take into account whether words were used earlier in the sentence! While this simple transition matrix approach can auto-complete some sentences, it's far from sufficient and doesn't come close to being state-of-the-art.

Can you devise other ways to decompose or "tokenize" our data set? What kinds of data sets can you come up with that can help correct the auto-complete in the above examples? What other limitations to this method do you foresee?

Drawbacks and Limitations

It's essential to understand the fundamental components of NLP and machine learning. While tools in NLP continue to improve, it should be noted that even state-of-the-art methods are not without flaws. For example, if I prompt the chatGPT program with "Is 873 a prime number?" it returns nonsensical statements such as "*If we divide 873 by 3, we get 873 / 3 = 291. Since 873 is not divisible by 3, it is not a multiple of 3.*" and "*If we divide 873 by 5, we get 873 / 5 = 174.*" As another example, if we take a closer look at Figure 1, we might notice that the princess's hands look a little strange and don't fit the image well. These technologies and tools also present digital-trust concerns: will we be able to tell whether someone wrote a paragraph themselves or whether it was written by a machine learning algorithm? Can you differentiate a human's writing from AI?[5] What about images where human faces are AI-generated? Can we tell who is a real human in a photo and who is not? There are also ethical concerns in developing such machine learning tools. For example, if a company uses artists' images to train their algorithm, who owns the rights to the generated images? Can machine learning algorithms be biased against protected groups of people? Understanding how machine learning algorithms and models are trained and what data is used to create them helps us understand these tools' weaknesses or limitations.

So what does all of this really mean to us? With great math comes great responsibility!

[5] To demonstrate how difficult this task can be, I snuck one extra paragraph written by chatGPT into this article! Can you figure out which one it is?

Romping Through the Rationals, Part 2

by Ken Fan | edited by Jennifer Sidney

Emily and Jasmine are studying the sequence obtained by starting with 0, then successively applying the function $f(x) = 1/(1 + \lfloor x \rfloor - \{x\})$ to its terms. A social media post Jasmine read claims that every nonnegative rational number occurs in the sequence exactly once.

Jasmine: What are you seeing?

Emily: Actually, it's nothing.

Jasmine: Now I'm really curious!

Emily: Well, I was looking at the numbers in the sequence that we've computed so far, and it seems like the denominator of each fraction becomes the numerator of the next fraction.

Jasmine looks over the sequence, which begins 0, 1, 1/2, 2, 1/3, 3/2, 2/3, 3, 1/4, 4/3, ….

Jasmine: Wow, I didn't notice that before! That's amazing! Why do you say that it's "nothing"?

Emily: Because we've already proven it!

Jasmine: We did?

Emily: I think so. For a rational number P/Q in lowest terms, we computed that $f(P/Q) = Q/(QK + Q - R)$, where K is the quotient and R is the remainder obtained when you divide P by Q. When we found this formula, we assumed that $Q > 1$. But the formula is valid even when $Q = 1$; in that case, $K = P$ and $R = 0$, so the formula gives us $Q/(QP + Q - 0) = 1/(P + 1)$, which we know is $f(P/1)$. We also showed that Q and $QK + Q - R$ are relatively prime, so the fraction $Q/(QK + Q - R)$ is in lowest terms. Thus the numerator of $f(P/Q)$ is Q, and that is the denominator of P/Q.

Jasmine: Gosh, it was staring at us in the face! I didn't notice it until you pointed it out. I was so concerned about showing the reciprocation property that I completely missed the denominator becoming the numerator. That observation has to be of some value. How curious! If the claim about getting all the nonnegative rational numbers is true, then the sequence of numerators would be an integer sequence with the property that the ratios of each term to the next term would produce every single nonnegative rational number exactly once. That seems like a peculiar property for a sequence. Are such sequences common?

Emily: Wow, you're right! Let's think about that for a bit.

Jasmine: Okay!

Emily: So you're interested in sequences of integers, a_1, a_2, a_3, …, that have the property that consecutive ratios a_k/a_{k+1} produce every nonnegative rational number exactly once. Maybe we should add a requirement that consecutive terms be relatively prime so that all the ratios are automatically in lowest terms. If that condition is too limiting, we can always drop it later.

Jasmine: Sounds fine to me. One thing's certain – such a sequence must start with 0; to get 0, some term would have to be 0, and if the 0 appears after the first term, then we'd end up dividing by 0, which is bad. And if the sequence starts with 0, then the next term would have to be 1 to suit the condition that consecutive numbers are relatively prime. So all such sequences begin 0, 1, ….

As often happens when "talking math," certain ambiguous terms acquire definite meaning only in context. For this conversation, when Emily and Jasmine refer to "ratios of consecutive terms" in a sequence, they mean the ratios of each term to the next term (as opposed to the ratios of each term to the previous term).

Emily: Hmm, then what?

Jasmine: It seems like the third term could be *any* positive integer.

Emily: Okay, so let's say P is a positive integer and the sequence begins 0, 1, P, …. After the P, it looks like the only numbers that we'd have to avoid are numbers that aren't relatively prime to P. So we can pick any positive integer Q that is relatively prime to P to extend the sequence. Then after the Q, we can again pick any positive integer relatively prime to Q, unless we happened to choose 1 for Q; in that case, we would also need to avoid P so that we don't get the number $1/P$ twice.

Jasmine: I see. As we build the sequence, we can use any positive integer relatively prime to the last number, provided that the new ratio does not duplicate a ratio that occurred earlier. But there are infinitely many positive integers that are relatively prime to any given positive integer and only finitely many consecutive ratios of terms that have occurred, so it's always possible to extend the sequence. I guess the problem isn't producing a sequence whose consecutive terms are *distinct*. The challenge is to ensure that *every* nonnegative rational number appears as a ratio of consecutive terms.

Emily: Suppose we just go ahead and build some sequence. If ratios of consecutive terms don't yield all nonnegative rational numbers, could we somehow modify the sequence to get the missing numbers?

Emily and Jasmine try to imagine ways of modifying an existing sequence so that it captures missing rational numbers.

Jasmine: I can't think of a way to do that.

Emily: I'm curious. What if we try to build such a sequence in some greedy way? For example, we can use "greedy" to mean that we extend the sequence by the *smallest* positive integer that doesn't produce a duplicate rational number. That sequence will be unique. Would it yield all the nonnegative rational numbers as ratios of consecutive terms?

Jasmine: That sounds like fun. Let's build the sequence!

Emily: Okay. So the sequence, like all such sequences, will begin 0, 1. The smallest positive integer that can come next would be 1, because so far, only the rational number 0/1 occurs as a ratio of consecutive terms. So the first three terms are 0, 1, 1.

Jasmine: We can't put 1 again, but we can put 2, so 0, 1, 1, 2.

Emily: And then comes another 1: 0, 1, 1, 2, 1. So far, the rational numbers 0, 1, 1/2, and 2 appear as ratios of consecutive terms.

Jasmine: Three comes next: 0, 1, 1, 2, 1, 3.

Emily: And then comes yet another 1: 0, 1, 1, 2, 1, 3, 1.

Emily and Jasmine continue, extending their sequence out to 12 terms:

0, 1, 1, 2, 1, 3, 1, 4, 1, 5, 1, 6, ….

Jasmine: How about that? After the initial 0, the list is just the sequence of positive integers, in increasing order, with 1's inserted between consecutive numbers!

Emily: So, unfortunately, if we use this greedy algorithm, only integers and reciprocals of integers will appear as ratios of consecutive terms. We'll never get 2/3.

Jasmine: All this just makes me think the social media fact, if true, is all the more remarkable!

Emily: I want to try one more way to build up a special sequence. I think part of the problem is that the greedy algorithm loves the number 1 – the smallest positive integer – so we get stuck seeing only fractions with a numerator or denominator of 1. So what if we try to prevent this by adding the requirement that we avoid positive integers greater than n until *all* fractions whose numerator and denominator (in lowest terms) are both less than or equal to n have appeared as a ratio of consecutive terms, unless we're absolutely forced to break this requirement. When all such fractions are accounted for (as ratios of consecutive terms), we increase the cap on the size of the numerator and denominator by 1.

Jasmine: Let's try!

Emily: As always, the sequence begins with 0, 1. And I guess we'll start our cap at 1. We haven't gotten 1 as a ratio of consecutive terms yet, so 1 comes next: 0, 1, 1. And now we have gotten all the rational numbers whose numerator and denominator are less than or equal to 1, so we increase our cap to 2.

Jasmine: The sequence continues: 0, 1, 1, 2, 1. Now all rational numbers with numerators and denominators less than or equal to 2 occur, so we increase our cap to 3.

Emily: And it continues: 0, 1, 1, 2, 1, 3, 1. Here, 1, 2, and 3 are all not good, because they would replicate an earlier ratio. So we have no choice but to put 4, even though not all fractions with numerators and denominators less than or equal to 3 have appeared. So we have to go 0, 1, 1, 2, 1, 3, 1, 4. Then 1 comes next, then … rats! This new rule doesn't change anything! We're going to get the exact same sequence we got without the rule. Maybe we should go back to thinking about the original problem; if we can solve that, then at least we'd have one example of a special sequence of the type we're now trying to build.

Jasmine: Actually, I like what you're trying to do in creating a rule to uniquely specify a sequence. I feel like some rule should work. For each new term, you were seeking the smallest positive integer that works; that, unfortunately, led to a sequence that produces only integers or reciprocals of integers as ratios of consecutive terms. As you said, that seems to happen because 1 is the smallest positive integer. What you're effectively doing is specifying, for each positive integer n, a unique order to all the numbers that are relatively prime to n. Each time n occurs in the sequence, you are following that occurrence with the next number in that unique order. The order you've chosen is the natural order on positive integers, going from the smallest, which is 1, to infinity. What if we change this order by going down from n instead of up from 1? When we get to 1, we then continue with $n + 1$ and start going up? Perhaps with this ordering, more rational numbers will occur as ratios of consecutive terms.

Emily: I like that! Let's try it! As always, the sequence begins 0, 1. Starting at 1 and going down, the next number we should try first is 1. Because 1/1 = 1 has not yet occurred as a ratio of consecutive terms, the next term is 1: 0, 1, 1. Given that we have another 1, we go to the next number relatively prime to 1 in the order you described. Since we just did the 1 and there's no positive integer below that, we would try 2; 2 is good, so the sequence continues: 0, 1, 1, 2. For the number 2, the numbers relatively prime to 2 are the odd numbers. Going down from 2, the first odd number we encounter is 1. Since 2/1 has not yet occurred as a ratio of consecutive terms, the next term is 1: 0, 1, 1, 2, 1. After this third 1, the next number up, according to your ordering scheme, is 3, since we already put a 2 after a 1. So it continues: 0, 1, 1, 2, 1, 3. Working down from 3, the first number we encounter that is relatively prime to 3 is 2, and 3/2 hasn't occurred yet, so we have 0, 1, 1, 2, 1, 3, 2. Yay! We get a number (3/2) that isn't an integer or the reciprocal of an integer!

Jasmine: Then 3 comes next, because we already put a 1 after the first 2 in the sequence, and you can't go down from 1 so we start going up from 2: 0, 1, 1, 2, 1, 3, 2, 3. Then 4, right?

Emily: Actually, I think 1 comes next, because the numbers that are relatively prime to 3 should occur in the order 2, 1, 4, 5, 7, etc., as per your scheme, and 1 has not occurred after 3 yet.

Jasmine: Oh, you're right! So the sequence continues 0, 1, 1, 2, 1, 3, 2, 3, 1. *Now* comes 4!

Emily: Yes, 4 is up next.

Emily and Jasmine continue computing terms and find

0, 1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 4, 1, 5, 4, 5, 3, 5, 2, 5, 1, 6, 5, 6, 1, 7, 6, 7, 5, 7, 4, 7, 3, 7, 2, 7, 1, …

Emily and Jasmine examine the sequence.

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n-Cubes and Up Operators

by Robert Donley[1]
edited by Amanda Galtman

In the previous installment of our path counting series, we developed the basic theory of graded posets and described the important example of a Boolean poset. We take a closer look at the geometry of these posets in terms of higher-dimensional cubes, and we find yet another formulation of the machinery underlying Pascal's triangle through "up" operators.

To recall, the elements of the Boolean poset B_n are n-tuples with entries 0 or 1, and the partial ordering is defined by entry-wise comparison: $y \leq z$ if and only if every entry of the n-tuple y is less than or equal to the corresponding entry of the n-tuple z. See Figure 1 for the Hasse diagrams of B_2 and B_3.

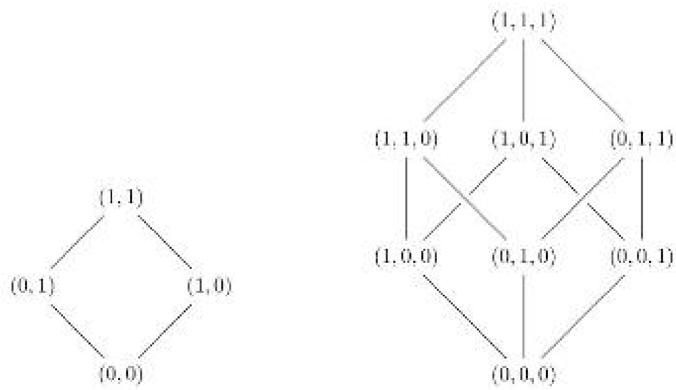

Figure 1. The Hasse diagrams for B_2 (left) and B_3 (right).

Both examples appear cubic in nature, and we shall build general n-dimensional cubes using Boolean posets as a guide. In turn, cubes provide further information about Boolean posets.

Definition. The n-cube $\boldsymbol{C_n}$ is the set of all n-tuples $(x_1, ..., x_n)$ of real numbers such that $0 \leq x_i \leq 1$ for each i. The **vertices** of C_n are those points with each x_i equal to either 0 or 1. Thus we identify the vertices of C_n with the elements of B_n.

(Since no Catalan numbers appear in this installment, we'll use the notation C_n for the n-cube.)

Figure 1 suggests a method for constructing C_{n+1} from C_n. For each element of C_n, we append a coordinate x_{n+1} to obtain an $(n + 1)$-tuple. Notice that for any c between 0 and 1, inclusive, the points $(x_1, x_2, ..., x_{n+1})$ in C_{n+1} with $x_{n+1} = c$ are also n-cubes. We'll denote these n-cubes with the notation $C_{n,c}$. The edges of C_{n+1} consist of all the edges of $C_{n,0}$ and $C_{n,1}$ as well as edges obtained by connecting each vertex in $C_{n,0}$ to the corresponding vertex in $C_{n,1}$.

Exercise: Construct the square C_2 in this manner from the interval $C_1 = [0, 1]$. Identify $C_{1,0}$, $C_{1,1}$, and $C_{1,1/2}$. Repeat this construction to get C_3 from C_2.

Exercise: Choose a coordinate x_i in C_n. Prove that the set of points with $x_i = 0$ is an $(n - 1)$-cube. Note that this result holds for all $x_i = c$ for any constant c such that $0 \leq c \leq 1$.

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[1] This content is supported in part by a grant from MathWorks.

The previous installment asked you to draw the Hasse diagram of B_4. Consider the planar representation of the 3-cube C_3 in Figure 2 below.

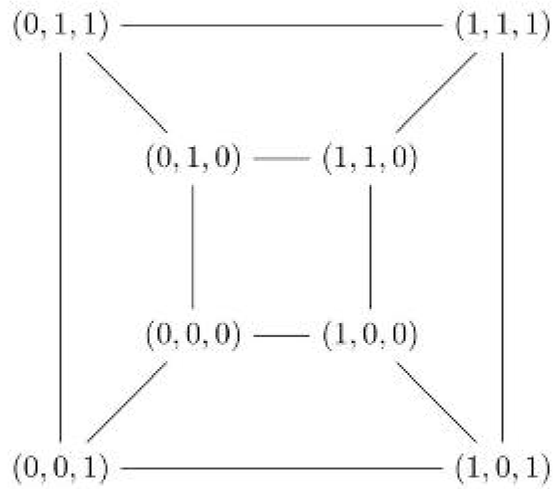

Figure 2.

Exercise: Using the construction of 4-cube C_4 (a.k.a. the **hypercube)** from C_3, extend this planar representation of C_3 to a planar representation of C_4 and label each vertex with its coordinates. Can you see how your diagram for C_4 corresponds to the Hasse diagram for B_4? Verify that the numbers of edges in C_4 and B_4 agree.

Let's count the number of nodes and links in the Hasse diagram of B_n using the n-cube. Denote the number of vertices and edges in C_n by V_n and E_n, respectively. The number of vertices in C_n is $V_n = 2^n$. For the interval, $V_1 = 2$, and we obtain successive counts by doubling the previous count. For edges, we shall derive the explicit formula

$$E_n = n2^{n-1}.$$

Exercise: Verify that the formula for E_n gives the correct values when $n = 1, 2, 3$, or 4.

When constructing C_{n+1}, we noted that edges of C_{n+1} consist of all the edges of $C_{n,0}$ and $C_{n,1}$ as well as edges connecting a vertex of $C_{n,0}$ with the corresponding vertex of $C_{n,1}$. That is, $E_{n+1} = 2E_n + V_n$.

Exercise: Verify that the explicit formula for E_n satisfies the recursive formula $E_{n+1} = 2E_n + V_n$.

Exercise: Prove the identity

$$1 + 2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 1.$$

Exercise: Derive the formula for E_n using the recursive formula and the preceding identity.

Since the formula for E_n is a product, we might guess that there's a proof involving a clever use of the Matching Rule. See if you can think of one before reading any further. Hint: We're going to show one way by exploiting the symmetry of the cube.

Definition. The **Hamming distance** between any two elements of B_n is given by the number of unequal entries. We use this terminology interchangeably for nodes in B_n and vertices in C_n.

For instance, the Hamming distance between (1, 0, 1) and (1, 1, 0) is 2. (By contrast, the distance formula shows that the Euclidean distance between (1, 0, 1) and (1, 1, 0) is $\sqrt{2}$.)

Consider the vertex (0, …, 0) in C_n. For $n = 1$, 2, or 3, we see directly that there are n edges with (0, …, 0) as an endpoint. If we suppose this is generally true for C_n, then, in the construction of C_{n+1}, the only edge linking (0, …, 0) to $C_{n,1}$ contains (0, …, 0, 1). That is, each iteration of the construction adds one new edge containing (0, …, 0).

Exercise: Prove that the edges that have a vertex v in C_n as an endpoint are exactly those edges for which the other vertex is at Hamming distance 1 from v. Explain why there are n such edges.

To apply the Matching Rule, observe that the total number of edges equals the number of pairs of vertices that are Hamming distance 1 apart. The formula for E_n follows: to construct an edge, we choose one of the 2^n vertices and then choose one of its adjacent vertices (at distance 1). Since each edge contains two vertices, every edge gets counted twice, so we divide the product by 2.

The Hamming distance condition for adjacent vertices is reflected directly in B_n. If two vertices x and y differ in exactly one entry, then either x covers y or y covers x in B_n. Each link in the Hasse diagram corresponds to a covering relation in B_n, so, with respect to the grading, x and y belong to adjacent levels in the Hasse diagram.

To count the number of links between rank k and rank $k + 1$ in B_n, recall that there are $C(n, k)$ elements at level k. An element with k ones, such as (1, …, 1, 0, …, 0), is covered by $n - k$ elements with rank $k + 1$. Thus there are $(n - k)\ C(n, k)$ such links in B_n. Summing over all levels in B_n gives the formula for $E_n = n\ C(n, 0) + (n - 1)\ C(n, 1) + \ldots + C(n, n - 1)$. Equating this to our earlier formula, we find

$$n\ C(n, 0) + (n - 1)\ C(n, 1) + \ldots + C(n, n - 1) = n2^{n-1}.$$

Exercise: Verify this formula for $n = 1$, 2, 3, and 4.

These ideas allow us to organize the covering relations in a way that enables efficient lattice path counting in graded posets. First, we describe some notions from linear algebra, one of the most powerful and applicable tools in all of mathematics.

Definition. If X is any finite set, we define a **linear combination** of elements in X to be a formal sum

$$c_1x_1 + \ldots + c_nx_n$$

where each c_i is a real number and each x_i is in X. One way to think of this is that we associate to each element x_i in the sum the real number c_i; elements in X that are not used explicitly in the sum are associated to the real number zero.

Linear combinations admit two operations: addition (which takes two linear combinations and returns a third) and scalar multiplication (which takes a real number and a linear combination and returns a linear combination). Let a, c_i, and d_i be real numbers. The sum of $c_1x_1 + ... + c_nx_n$ and $d_1x_1 + . . . + d_nx_n$ is given by the formula

$$(c_1x_1 + ... + c_nx_n) + (d_1x_1 + . . . + d_nx_n) = (c_1 + d_1)x_1 + . . . + (c_n + d_n)x_n,$$

and scalar multiplication of $c_1x_1 + ... + c_nx_n$ by the real number a is given by the formula

$$a(c_1x_1 + ... + c_nx_n) = ac_1x_1 + ... + ac_nx_n.$$

Definition. A **linear operator** T is a map that sends the set of linear combinations on X to that same set and satisfies $T(cu) = cT(u)$ and $T(u + v) = T(u) + T(v)$ for any linear combinations u, v and real number c.

In other words, T sends linear combinations to linear combinations in a way that respects addition and scalar multiplication. Note that the rules for a linear operator imply

$$T(c_1x_1 + ... + c_nx_n) = c_1T(x_1) + . . . + c_nT(x_n).$$

That is, if we have a linear operator T, its value on any linear combination is entirely determined by the values on each x_i. Conversely, to define a linear operator, it suffices to specify which linear combination each x_i is mapped to.

Definition. For the Boolean poset B_n, we define the **up** linear operator U for linear combinations on B_n. To define $U(v)$ for an element v in B_n, we sum the elements that cover v. In other words, the terms of this sum are the elements obtained by replacing a single 0 in v with a 1. To figure out what U maps general linear combinations of elements of B_n to, we use the linearity rules stated in the definition of a linear operator above.

For instance, in B_3, $U((1, 0, 0)) = (1, 1, 0) + (1, 0, 1)$.

On the other hand, we record the following iterations of U on $(0, 0, 0)$ in B_3:

$$\begin{aligned} b_1 &= U((0, 0, 0)) = (1, 0, 0) + (0, 1, 0) + (0, 0, 1) \\ b_2 &= U(b_1) = 2(1, 1, 0) + 2(1, 0, 1) + 2(0, 1, 1) \\ b_3 &= U(b_2) = 6(1, 1, 1). \end{aligned}$$

In more detail, here's how we computed $U(b_1)$:

$$\begin{aligned} U(b_1) &= U((1, 0, 0) + (0, 1, 0) + (0, 0, 1)) \\ &= U((1, 0, 0)) + U((0, 1, 0)) + U((0, 0, 1)) \\ &= ((1, 1, 0) + (1, 0, 1)) + ((1, 1, 0) + (0, 1, 1)) + ((1, 0, 1) + (0, 1, 1)) \\ &= 2(1, 1, 0) + 2(1, 0, 1) + 2(0, 1, 1). \end{aligned}$$

The up operator generalizes Pascal's recurrence for Pascal's triangle. If we apply U to a linear combination $v = c_1x_1 + ... + c_nx_n$, the coefficient of x_i in $U(v)$ is the sum of all c_j for which x_i covers x_j. Starting from the minimum $(0, 0, 0)$, repeated application of U records chain counts from $(0, 0, 0)$ to other elements in B_3. If we iteratively apply U to $(0, 0, 0)$ a total of k times, all the elements in the resulting linear combination will have rank k.

From the previous installment, the number of maximal chains in B_n from $(0, \ldots, 0)$ to an element of rank k is $k!$. The calculation above confirms the case of B_3.

Exercise: Repeat the maximal chain count calculations for the up operator U on B_4.

Much of what we have done to this point generalizes immediately from cubes to other graded posets. For the remainder of this article, we recast earlier results in this series on Pascal's triangle using the up operator.

Example: Consider the rectangular grid starting at $(0, 0)$ with far corner (m, n). The up operator in this case sends (x, y) to $U((x, y)) = (x, y + 1) + (x + 1, y)$. If $x + 1$ exceeds m or $y + 1$ exceeds m, we omit that element from the result.

When $m = 3$ and $n = 2$, we have

$$\begin{aligned} b_1 &= U((0,0)) &&= (0, 1) + (1, 0) \\ b_2 &= U(b_1) &&= (0, 2) + 2(1, 1) + (2, 0) \\ b_3 &= U(b_2) &&= 3(1, 2) + 3(2, 1) + (3, 0) \\ b_4 &= U(b_3) &&= 6(2, 2) + 4(3, 1) \\ b_5 &= U(b_4) &&= 10(3, 2) \end{aligned}$$

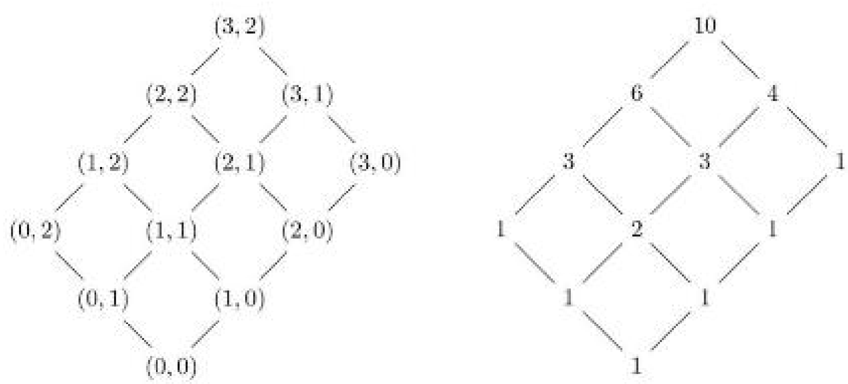

Figure 3.

Exercise: Verify the chain counts in Figure 3 using the up operator.

For Pascal's triangle, another way to view the up operator is in terms of polynomials in x and y. Now the entries in an ordered pair correspond to the exponents on x and y. Instead of using rules for linear combinations, we use operations on polynomials to perform the necessary algebra.

The up operator takes the form $U(p(x, y)) = (x + y)p(x, y)$. If the polynomial $m(x, y) = 1$ corresponds to the minimum $(0, 0)$, then iteratively applying U to $m(x, y)$ a total of k times yields the polynomial $(x + y)^k$, confirming the binomial theorem.

Exercise: Extend the development of the up operator above to ordered triples (m, n, p) of nonnegative integers. How does the analogue of Pascal's triangle look? Describe the corresponding Pascal's recurrence and trinomial theorem. Calculate the lattice path counts for elements up to rank 3 using the up operator. (Hint: Some help is given in the previous article for lattice path counting in a rectangular box.)

Learn by Doing

Exponentials

by Lightning Factorial | edited by Anne Paoletti

Note: Problems vary in level of involvement. You'll find that you can do some quickly, but others (in red) are intended to give you something to think about for days. You're always welcome to email us or ask a mentor about this at the club.

Let's learn about exponentials.

To set the right mood, take a moment to get comfortable and relax. We're going to get this right, and to do things right in math, it's important to go step by step and not rush. Take your time.

Multiplication

The concept of exponentials is built using the concept of multiplication. A solid understanding of multiplication is important to have before learning about exponentials. Since multiplication isn't the topic of this "Learn by Doing," we won't discuss it in detail, so if you aren't yet comfortable with multiplication, please think about that first (or ask about it at the club!), then return here. This article won't go anywhere!

Let's recall one fact about multiplication: multiplication was initially introduced as a shorthand for repeated addition and only for whole numbers. For example, "12 times 5" is shorthand for adding up twelve 5's: $5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5$.

As with all mathematical notation, the notation is used because people find it helpful. Adding the same number repeatedly occurs so frequently that the notation for multiplication is most welcome. Here are just a few problems that call for repeatedly adding the same number:

1. Maribelle donates $15 each month to Girls' Angle. How much money will she have donated in 48 months?

2. The seats in an auditorium are arranged in a rectangle with 72 seats in each row. If there are 40 rows of seats, how many total seats are there in the auditorium?

3. You're organizing a trip to an amusement park for 17 people. You need to get each person 8 ride tickets for the trip. How many ride tickets do you need to purchase?

After being defined for whole numbers, people later extended multiplication so that any two numbers, whole or not, may be multiplied together. This extension to other numbers is cleverly carried out so that the basic properties of multiplication (the commutative, associative, and distributive properties) remain valid. If you're unfamiliar with these properties, please look them up or ask about them at the club. You don't have to review them now, however.[1]

The reason we mentioned all this about multiplication here is because we'll soon see that exponentials followed an analogous development. We'll see that exponents were also initially defined for whole numbers, and then later, extended so that the exponent can be any real number. And the way this extension was accomplished was by insisting that certain rules persisted, just like with the extension of multiplication to all real numbers.

[1] For a detailed account of how multiplication is extended to negative numbers, check out the article *Negative Times Negative Is Positive* on pages 19-21 of Volume 3, Number 3 of the Girls' Angle Bulletin.

From repeated addition to repeated multiplication

4. The number of bacteria in a petri dish doubles every day. If you start with 1 bacterium, how many bacteria will there be in 14 days?

Since the number of bacteria doubles every day, to find out how many bacteria there will be tomorrow, we have to multiply the number of bacteria there are today by 2. Starting with 1 bacterium, we have to do this 14 times: $1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$.

This is our first example of an **exponential** formula.

5. What notation would *you* invent to denote $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$ so that you don't have to write down fourteen 2's and thirteen ×'s?

Please share it with us; we'd love to hear what you come up with! There are infinitely many notations one might come up with to represent this huge product. In fact, there was a long period of several decades where many different notations were in use.[2] One such notation was to write our product like this: $\boxed{14}2$. The number of times one repeats the multiplication by 2 was put inside a box preceding the 2. But that notation did not get widely adopted.

Through the years, the notation has settled down to the following: 2^{14}. The number being multiplied is adorned with a superscript to its upper right which tells how many times you repeat the multiplication. The number that is repeatedly multiplied (the 2) is called the **base** of the exponential. The superscript tells how many times you repeat the multiplication (the 14) and is called the **exponent** or the **power**. The exponential expression 2^{14} can be read: "two to the fourteenth power," "two the fourteenth," or "two to the power fourteen." (It may also be read "two times two times two times two times two times two times two times two times two times two times two times two times two times two," although that certainly defeats the purpose!) There's also special terminology for small exponents: 5^2 can also be read as "5 **squared**" and 6^3 can also be read as "6 **cubed**."

Today, just about everybody uses the superscript notation for exponents, so whether we like it or not, we should learn it for the sake of communication. (Warning: superscripts do not always denote exponents, so be mindful of the context!) The best way to learn notation is to use it. So...

6. Express these products using exponential notation.

A. $7 \times 7 \times 7 \times 7 \times 7 \times 7$ B. $3 \times 3 \times 3 \times 3$ C. $8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8$

For answers, see page 29.

Now answer the following questions by giving your answer using exponential notation. You don't need to carry out the multiplications that your exponential expressions describe.

7. Massachusetts license plates consist of 6 capital letters or numbers. How many different license plates are possible in that state?

[2] See *A History of Mathematical Notations, Volume 1* by Florian Cajori.

8. In the Chinese art of pulled noodles, a chef stretches and folds over a wad of dough multiple times to make noodles. Each stretch and fold doubles the number of strands. How many strands of noodles will there be after stretching and folding 20 times?

Isolating the exponential from the computation of the exponential

The intended answers to the last two questions are 36^6 and 2^{20}, respectively.

You might feel that these problems are not yet fully solved if one leaves the answers in the form of an exponential expression. After all, 36^6 is not a single decimal number – it is an expression that describes a number by telling us what we have to do to get that number. (If you must know, if we carry out the multiplication, we get 2,176,782,336... more than adequate to suit Massachusetts' needs for the foreseeable future!)

You might even feel that exponentials are "hard" because it's hard to multiply large numbers together accurately without a calculator, and computing an exponential often requires us to multiply very big numbers together. Not only are there many chances to make a computational error, the computation itself is tedious.

If this is true of you, it's important to realize that what you are describing as "hard" is not the *concept* of exponentials, but rather, the *computation* of them. And, I have some good news for you: it is often acceptable to simply leave a number expressed as an exponential. Doing so is actually *more* informative because it tells you how a number is obtained. Note, for example, that it is not easy to see that the number 2,176,782,336 is $36 \times 36 \times 36 \times 36 \times 36 \times 36$. There's even a notation that many scientists use called **scientific notation** in which quantities are expressed using explicit exponential notation.

Of course, if you're taking a test, be sure to give the answer as the test requests.

Leaving a number in exponential form as opposed to "simplifying it" by carrying out the multiplications is quite similar to the use of fractions. For example, what is 3 divided by 7? Well, we could perform a long division and write this out as the repeating decimal $0.\overline{428571}$. However, that's a bit tedious. We *understand* what 3 divided by 7 means – it is the number that, if you repeatedly add it 7 times, you get 3. And, in fact, our understanding of what it means may well be more important than knowing its decimal representation, and for this reason, people are accustomed to writing that 3 divided by 7 is 3/7. Think carefully about what's going on here. Doesn't it feel like cheating to give "3/7" as the answer to the question "What is 3 divided by 7?" It's like answering the question, "How many stars are there in the sky?" with "The number of stars in the sky is the number of stars in the sky." The only thing we have done is to replace "divided by" with a symbol that stands for "divided by!" Yet, we're accustomed to thinking of and working with "3/7" as a number, not a division problem.

Just as we can accustom ourselves to regard 3/7 as a number (as well as a division problem), so too can we accustom ourselves to regarding 36^6 as a number (as well as a repeated multiplication problem). If you can free yourself from demanding that an answer be a single decimal number and grow comfortable working directly with descriptions of numbers, you will open the door to a deeper theoretical understanding of how things connect because you will think in terms of some kind of understanding of the number beyond its quantitative value.

When we regard a fraction as a number instead of as specifying a division problem, some computations require less work while others require more. With fractions, multiplication and division are quicker, while addition and subtraction become more involved. For example:

9. What is $3/7 \times 7/2$?

On the other hand:

10. What is $3/7 + 7/2$?

Similarly, with exponentials, multiplication and division are often quicker to handle if you leave a number in exponential form as opposed to multiplying everything out. The reason for this is that, often, when you multiply and divide numbers, it helps to know factors of the numbers. For example, which of the following two problems do you find requires less work to answer?

11. What is 17,484 divided by 47?

12. What is 372×47 divided by 47?

The second requires hardly any work at all because division by 47 undoes multiplication by 47. Yet, these are the same problem since $17{,}484 = 372 \times 47$.

Or, more to the point of exponentials, which do you find requires less work to answer?

13. What is 59,049 divided by 729?

14. What is 9^5 divided by 9^3?

Again, the second problem requires hardly any work, even though they are the same problem. That's because the second problem gives you information about factors. The exponential 9^5 means $9 \times 9 \times 9 \times 9 \times 9$. Since multiplication is commutative and associative, this product can be grouped as $(9 \times 9) \times (9 \times 9 \times 9)$, or $9^2 \times 9^3$. Since division by 9^3 undoes multiplication by 9^3, when we divide 9^5 by 9^3, we get 9^2.

Properties of exponentials

The last problem illustrates a general property of exponentials, namely, $b^x b^y = b^{x+y}$, where b is any number and x and y are whole numbers. This and the properties described in the following problems all follow from the basic definition of b^x as repeated multiplication by b a total of x times, and properties of multiplication.

For Problems 15-17, a and b are any numbers and x and y are whole numbers.

15. Convince yourself that $b^x b^y = b^{x+y}$.

16. Convince yourself that $(b^x)^y = b^{xy}$.

17. Convince yourself that $a^x b^x = (ab)^x$.

Extending exponents to all real numbers

Just as multiplication can be extended beyond whole numbers by insisting that the commutative, associative, and distributive properties remain valid, so too can exponents be extended beyond whole numbers by insisting that the basic property that $b^x b^y = b^{x+y}$ remain valid.

For Problems 18-21, assume that b is a positive real number.

18. (Extending exponents to 0 exponents) If we insist that $b^x b^y = b^{x+y}$, for all positive real numbers b and numbers x and y, show that b^0 must be 1.

If you are having trouble with this last problem, let's think together. We want $b^x b^y = b^{x+y}$ to be true. We know what b^x is for any positive integer x, but we do not know the value of b^0. Can we pick values of x and y in such a way that the identity $b^x b^y = b^{x+y}$ involves at least one occurrence of b raised to the 0 power, and all the other exponentials being b raised to a positive whole number? If that can be arranged, then we would be able to solve for b^0 in terms of things that have already been defined.

19. Play around with picking different values for x and y to see if you can do this.

One way to show that b^0 must be 1 is to substitute 0 for x and 1 for y in the identity $b^x b^y = b^{x+y}$. When we do this, we get $b^0 b^1 = b^{0+1}$. Since $0 + 1 = 1$, we can simplify the right side of this equation to b^1 and get $b^0 b^1 = b^1$. And since b^1 is just b, our equation can be written as $b^0 b = b$. If we divide both sides by b, we get $b^0 = 1$.

Here are two more opportunities to get the hang of this way of thinking:

20. (Extending exponents to negative exponents) If we insist that $b^x b^y = b^{x+y}$, for all positive real numbers b and numbers x and y, show that b^{-x} must be equal to $1/b^x$.

21. (Extending exponents to rational number exponents) If we insist that $b^x b^y = b^{x+y}$, for all positive real numbers b and numbers x and y, show that $b^{1/n}$ must be an nth root of b, where n is a whole number. For definiteness, let's take $b^{1/n}$ to be the positive, real nth root of b.

Problems 18-21 show how exponents are extended to include rational numbers.

22. Let x and y be rational numbers and $b > 1$. Show that $x < y$ if and only if $b^x < b^y$. How should this statement be modified if $b = 1$ or $0 < b < 1$?

Expanding exponents to all real numbers

Here's a challenge:

23. How would you extend exponents to all real numbers?

(Spoiler alert!) There are many ways to extend exponents to all real numbers, but one has proven to be enormously useful: Extend exponents to real numbers in such a way that if we regard x as a real variable and b as a fixed positive real number, then the function b^x is continuous. (Intuitively, this means that the graph of $y = b^x$ may be drawn without having to lift the pencil.)

Notes from the Club

These notes cover some of what happened at Girls' Angle meets. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

Session 31 - Meet 9
November 3, 2022

Mentors: Elisabeth Bullock, Jade Buckwalter, Tharini Padmagirisan, Kate Pearce, AnaMaria Perez, Laura Pierson, Jane Wang, Jing Wang, Muskan Yadav

If you've just learned about the derivative in calculus, here's a challenge for you that will help to prepare your mind for multivariable calculus: In single variable calculus, the graph of a function is a curve in a coordinate plane, and one can use the derivative to compute the slope of tangent lines to this curve. For functions of two variables, the graph will be a surface in coordinate space. At any given point on the graph, there will be a tangent plane (consisting of many tangent lines) to the surface. Can you determine the equation of the plane that is tangent to the graph of $f(x, y) = x^2 + y^2$ at the point (1, 2, 5)?

Session 31 - Meet 10
November 10, 2022

Mentors: Jade Buckwalter, Cecilia Esterman, Anne Gvozdjak, Jenny Kaufmann, Shauna Kwag, Hanna Mularczyk, Tharini Padmagirisan, Kate Pearce, AnaMaria Perez, Laura Pierson, Jane Wang, Jing Wang, Rebecca Whitman, Muskan Yadav

To learn exponentials well, like anything in math, you have to play around with the concept. See page 23.

Session 31 - Meet 11
November 17, 2022

Mentors: Jade Buckwalter, Cecilia Esterman, Anne Gvozdjak, Hanna Mularczyk, Tharini Padmagirisan, Kate Pearce, AnaMaria Perez, Laura Pierson, Jane Wang, Jing Wang, Rebecca Whitman, Muskan Yadav

Members Beatrice, Eleanor, and Sadie, working with student mentor Alina, dreamt up this wild nonlinear system of equations. Can you solve it and show that the solution is unique?

$$\frac{T}{7S} + \sqrt{4} = 100G + 7M + \tfrac{1}{32} \qquad \frac{T+2S}{16S} = \left(100G - \frac{32}{9}M\right)^2$$

$$T + M + S + G = \tfrac{991}{800} \qquad T + 2S - 1 = G - \tfrac{2}{100}$$

Session 31 - Meet 12
December 1, 2022

Mentors: Jade Buckwalter, Cecilia Esterman, Anne Gvozdjak, Shauna Kwag, Hanna Mularczyk, Kate Pearce, AnaMaria Perez, Laura Pierson, Emma Wang, Jane Wang, Jing Wang, Rebecca Whitman, Muskan Yadav

After years of pandemic, we've brought back our traditional end-of-session math collaboration! Congratulations to all members for cracking the safe!

Calendar

Session 31: (all dates in 2022)

September	8	Start of the thirty-first session!
	15	
	22	
	29	
October	6	
	13	
	20	
	27	
November	3	
	10	
	17	
	24	Thanksgiving - No meet
December	1	

Session 32 (tentative schedule): (all dates in 2023)

January	26	Start of the thirty-second session!
February	2	
	9	
	16	
	23	No meet
March	2	
	9	
	16	
	23	
	30	No meet
April	6	
	13	
	20	No meet
	27	
May	4	

Girls' Angle has run over 150 Math Collaborations. Math Collaborations are fun, fully collaborative, math events that can be adapted to a variety of group sizes and skill levels. We now have versions where all can participate remotely. We have now run four such "all-virtual" Math Collaboration. If interested, contact us at girlsangle@gmail.com. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes for small groups on a wide range of topics. For inquires, email: girlsangle@gmail.com.

Here are answers to Problem 6 of *Learn by Doing* on page 24.

6. A. 7^6 B. 3^4 C. 8^9

Girls' Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ____________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

The $50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- ☐ Enclosed is a check for $50 for a 1-year Girls' Angle Membership.
- ☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

Girls' Angle

A Math Club for Girls

Girls' Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is $20/meet for members and $30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle is based in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: ________________________

(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is \$216 for one session (12 meets)	☐ I will pay on a per meet basis at \$30/meet.
☐ I will pay on a per meet basis at \$20/meet.	☐ I'm including \$50 to become a member, and I have selected an item from the left.

☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: ___________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____