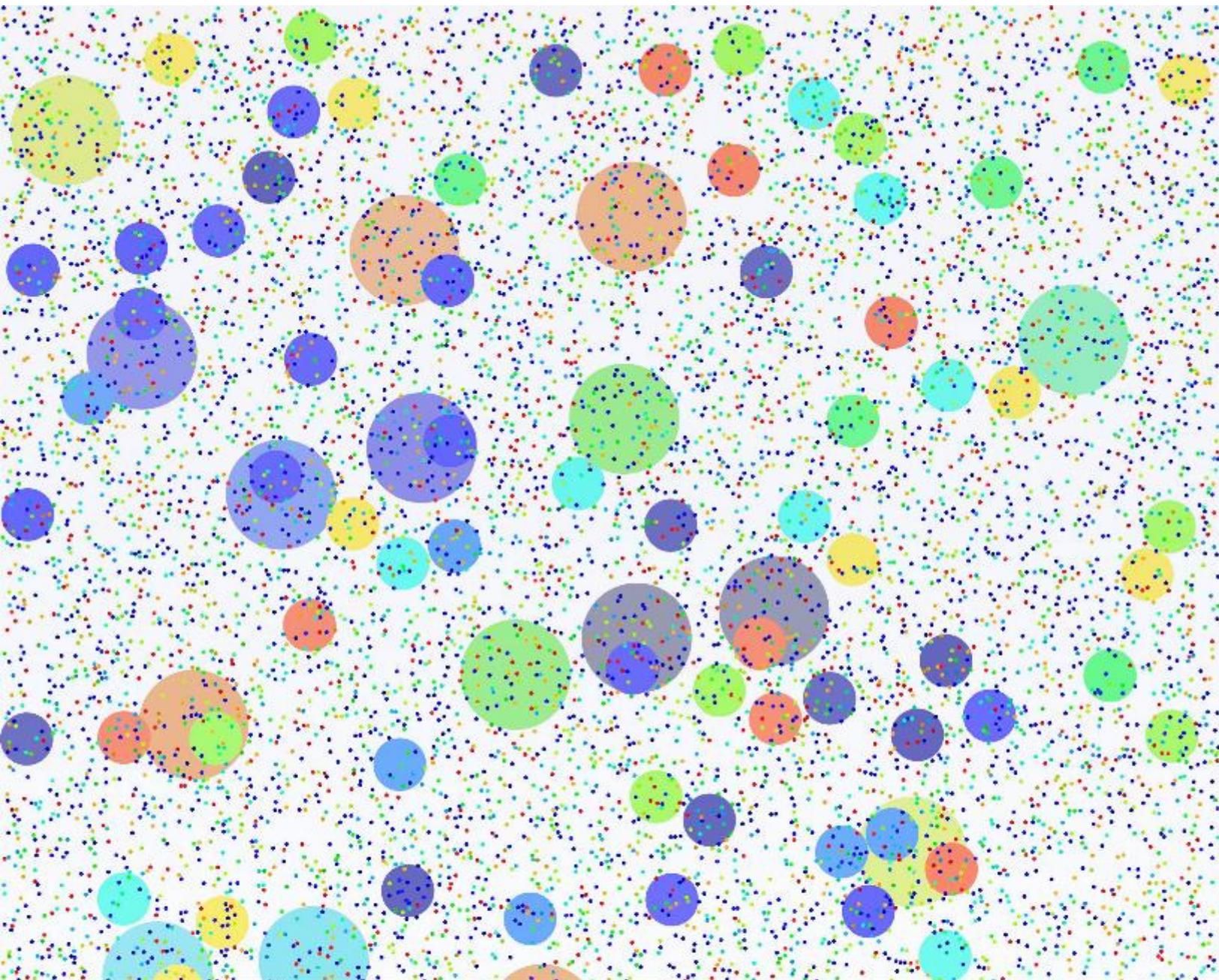


Girls' *Angle* Bulletin

June/July 2022 • Volume 15 • Number 5

To Foster and Nurture Girls' Interest in Mathematics



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From the Founder

I thank the organizers of the World Meeting of Women in Mathematics for putting on an excellent conference and for giving me the opportunity to present on what girls have taught me about how to teach math through the years. - Ken Fan, President and Founder

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Girls' Angle Bulletin

*The official magazine of
Girls' Angle: A Math Club for girls
Electronic Version (ISSN 2151-5743)*

Website: www.girlsangle.org
Email: girlsangle@gmail.com

This magazine is published six times a year by Girls' Angle to communicate with its members and to share ideas and information about mathematics.

Girls' Angle welcomes submissions that pertain to mathematics.

The print version of the Bulletin is printed by the American Mathematical Society.

Editors: Amanda Galtman
Jennifer Sidney
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The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

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On the cover: *Random Circles* by C. Kenneth Fan. See Problem 1 in AnaMaria Perez and Josh Josephy-Zack's Summer Fun problem set *Fermi Questions* on page 26.

An Interview with Zoi Rapti

Zoi Rapti is a Professor of Mathematics at the University of Illinois Urbana-Champaign. There, she is also a Professor at the Carl R. Woese Institute for Genomic Biology. She was born and raised in Greece. She earned her PhD in mathematics from the University of Massachusetts, Amherst.

Ken: One of the reasons we are so excited for the opportunity to interview you is because many of our members are interested in biology and biological applications of mathematics, and you've applied math to a number of fascinating biology topics, including the structure of DNA, modeling the spread of disease, and even the color patterns of bees. So, my first question, or perhaps more accurately, request, is: Could you say something about the relationship between math and biology. How does a mathematician view problems in biology? What is it about a problem in biology that makes you think, "there's math there"?

Zoi: Thank you very much for the opportunity for this interview. I have really enjoyed reading the bulletin.

The question about the relationship between math and biology is a great one. Unlike other disciplines, like physics and engineering, biology does not have as many "rules" and "laws." In my opinion, introducing mathematics into the biological sciences allows us to speculate about and test the existence of such rules and laws. Mathematical models are also a tool to test

...the process of becoming a professional mathematician is one full of challenges and self-doubt (article rejections, grant rejections, errors in proofs, and bugs in codes, to name a few)... It is the rare events in-between the many failures and challenges that make this profession worth pursuing.

biological hypotheses before embarking on their validation or rejection through time-consuming and costly experiments. For biologists interested in understanding the mechanisms driving biological processes and phenomena, mathematics is a valuable guide into what further experiments should be conducted and what data should be collected.

As a mathematician, I view problems in biology as puzzles. They are challenges where the goal is to use models and simulations to explain what my bright colleagues in the biological sciences observe. Together, we are looking for explanations.

My approach is that there is always math in biological problems. I try to select the ones where the mathematics is interesting and something novel can be found. Namely, I'm looking for problems or questions that have not been studied in this way before, but for which there is data and interest from biologists.

Ken: Could you please explain your work with bee color patterns and how math applies?¹ How did you become interested in this matter?

¹ See "Defining the colour pattern phenotype in bumble bees (*Bombus*): a new model for evo devo" by Rapti, Duennes, and Cameron, Volume 113, Issue

2, October 2014 of the *Biological Journal of the Linnean Society* on pages 384-404.

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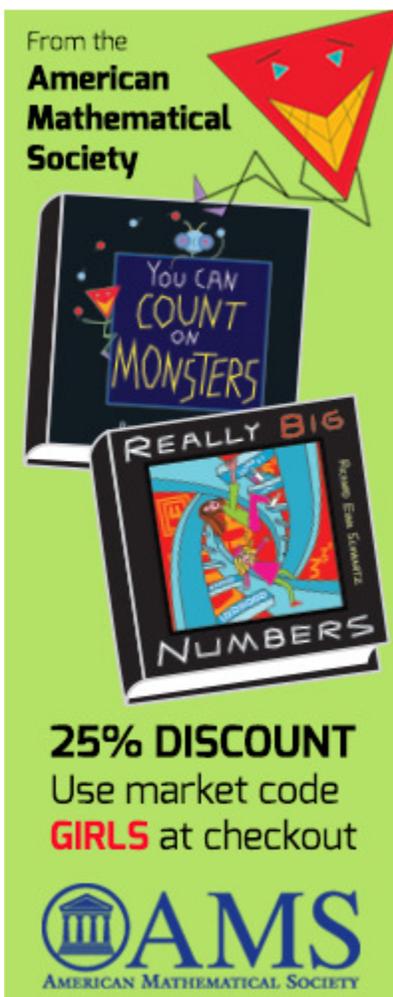
For this issue, those who do not subscribe to the print version will be missing out on a portion of this interview with Prof. Zoi Rapti and other content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

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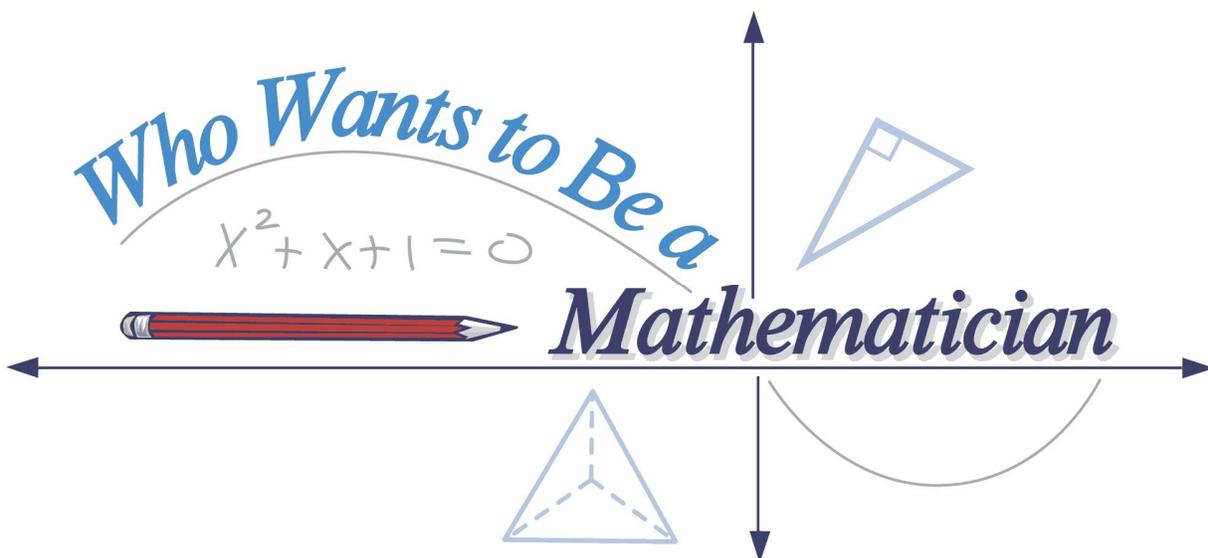
Thank you and best wishes,
Ken Fan
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Central Binomial Coefficients and Catalan Numbers

by Robert Donley¹

edited by Amanda Galtman

We continue our journey into the combinatorics of paths. In Part 1, *Shortcuts to Counting*, we considered different basic situations and formulas for counting; it will be helpful to review the Matching Rule, factorials, and combinations. In Part 2, we considered path counting in Pascal's triangle, Pascal's Identity, and, more generally, Chu-Vandermonde convolution.

Let's view paths in Pascal's triangle in another way. Suppose we start on the number line at position zero at time zero, and, as each second passes, we move one unit either to the left or to the right with equal chance, perhaps decided by flipping a fair coin. This is an example of a process we call a **random walk** (or, simply, **walk**). This process is equivalent to forming a path in Pascal's triangle: we form a word, reading left to right, by recording the move performed at the end of each second, and, at time N , a word of length N with letters L and R results.

Recall that the number of such words with K occurrences of the letter L is the combination $C(N, K)$, where K is between 0 and N , inclusive; for these walks, the ending position is $N - 2K$, a number between $-N$ and N . On the other hand, these combinations are the entries of the N th row of Pascal's triangle. A given combination $C(N, K)$ counts the number of paths to the K th entry in this row, allowing that the first entry represents $K = 0$.

Let's instead count **loops**, where a loop is a walk that returns to the starting position. Loops must have the same number of moves to the left as to the right, so $N = 2K$.

Definition. For $K \geq 0$, the **central binomial coefficient** is defined as $C(2K, K)$.

The central binomial coefficients are the numbers on the central vertical line of Pascal's triangle.

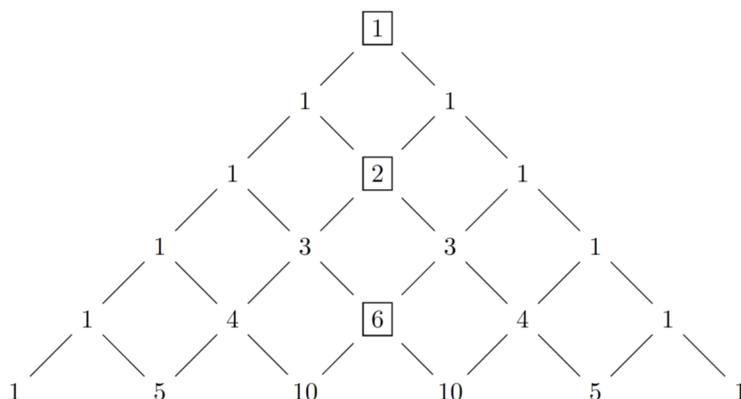


Figure 1. Highlighting the central vertical line of numbers in Pascal's triangle.

¹ This content is supported in part by a grant from MathWorks.

Exercise. Let P be a prime number strictly between K and $2K$. Explain why P must divide $C(2K, K)$.

On the other hand, central binomial coefficients play an important part in proofs of the Bertrand postulate, stated in rhyme by Paul Erdős:

“Chebyshev said it, and I say it again
There is always a prime between n and $2n$.”

(Here, $n > 1$.) This result requires advanced techniques to prove, but verifying directly up to $n = 25$ is a reasonable calculation by hand.

Exercise (Sieve of Eratosthenes). List the first 50 positive integers in a row and cross out the 1. In the list, find the first number that is neither crossed out nor circled. Call it n . Circle n and cross out all multiples of n in the list starting at $2n$. Repeat until all numbers are crossed out or circled. The circled numbers will be exactly the prime numbers less than 50. Verify that there is always a prime number between K and $2K$ for all $1 < K < 25$. (This method of producing the prime numbers in a list is known as the sieve of Eratosthenes.)

Exercise. Note that in the previous exercise, the last circled number for which we had to cross out a multiple that wasn't already crossed out was 7. For a list of $2K$ positive integers, can you guess the corresponding last circled number for which a multiple must be newly crossed out?

Exercise. Calculate and factor the first seven central binomial coefficients. Verify the “ P divides $C(2K, K)$ ” assertion for these cases.

Exercise. Find all loops counted by the central binomial coefficients with $K = 0, 1, 2, 3$, and 4. Also, find each corresponding word. For each $K \leq 7$, what fraction of walks of length $2K$ are loops? Recall that the number of walks of length N is 2^N .

Suppose our loops start at zero but take only nonnegative integers as positions; that is, imagine a wall that prevents us from moving to the left of zero. Our first thought might be that the number of these loops is half the corresponding central binomial coefficient. However, of the 20 loops of length six, only five loops avoid taking negative positions. The answer is far more involved and interesting — enter the **Catalan numbers**!

To develop the Catalan numbers, we revisit the techniques we used to understand combinations. Words for loops on the nonnegative integers, called **ballot sequences**, are characterized by two properties:

- There are an equal number K of letters L and R , and
- At any position in the word, the letter R appears as many or more times than the letter L up to that point.

Exercise. Describe what happens in an election corresponding to a ballot sequence.

Ballot sequences always begin with R and end with L . For instance, when $K = 1$, there is only the word RL . When $K = 2$, we have the words $RLRL$ and $RRLL$, and, for $K = 3$, we have $RRLLLL$, $RRLRL$, $RLRRL$, $RRLRL$, and $RLRLRL$.

Let's note some relationships here. For $K = 2$, the first case is just taking the case of $K = 1$ twice; that is, the corresponding loop returns to zero after two steps and begins again with only two steps left to use. The second case corresponds to a **proper loop**, i.e., one that cannot be split into smaller loops; it returns to zero exactly once. Removing the outermost letters of the corresponding ballot sequence results in the ballot sequence for $K = 1$.

Exercise. For $K = 3$, determine which loops are proper and which loops can be decomposed into smaller loops. For proper loops, verify that removing the outermost letters of the corresponding ballot sequence produces a ballot sequence for $K = 2$.

Exercise. Find all 14 words for the case $K = 4$. Determine which loops return to zero exactly once.

In our new situation, path counting with Pascal's Identity still works, but we must recalculate with values to the left of the center line set to zero. We call this triangle the **Catalan triangle**.

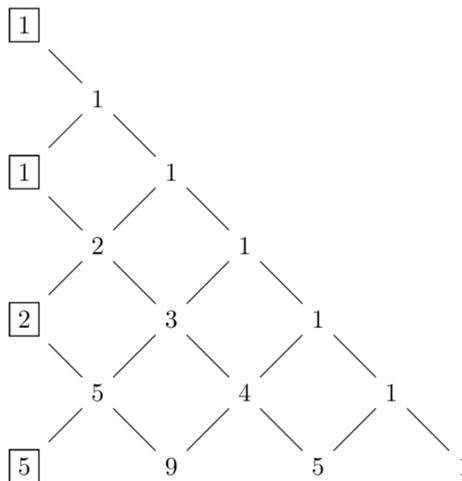


Figure 2. The Catalan triangle.

For $n \geq 0$, the n th **Catalan number**, denoted by C_n , is the entry on the leftmost vertical line in row $2n$ of the Catalan triangle. The Catalan number C_n counts the number of paths from the zeroeth row to this entry; such paths are called **Dyck paths**.

Exercise. The first five Catalan numbers are 1, 1, 2, 5, and 14. Find the next two values using the Catalan triangle.

Exercise. What do the other entries in the Catalan triangle count? (Hint: think of paths, words, and walks.)

In a trickier² construction of the Catalan triangle, you subtract Pascal's triangle from its shifted self. The goal is to close off the left-hand side of the triangle with a column of zeros in odd-

² From Ault and Kicey's book *Counting Lattice Paths Using Fourier Methods*, p. 12: "This type of extension..., prevalent throughout the signal processing literature, should not really be considered a *trick*, but rather a useful method..."

numbered rows. By the symmetry property of combinations, the middle entries in these rows of Pascal's triangle are equal. Thus, if we subtract such a row from its shifted self, a technique used for Pascal's Identity in Part 2, a zero occurs in the center of the new row. For example, using the third row of Pascal's triangle,

$$\begin{array}{cccc} -1 & -3 & -3 & -1 \\ & 1 & 3 & 3 & 1 \\ \hline -1 & -2 & 0 & 2 & 1 \end{array}$$

Alternatively, we repeat the construction for Pascal's triangle with starting entries -1 and 1 in the top row:

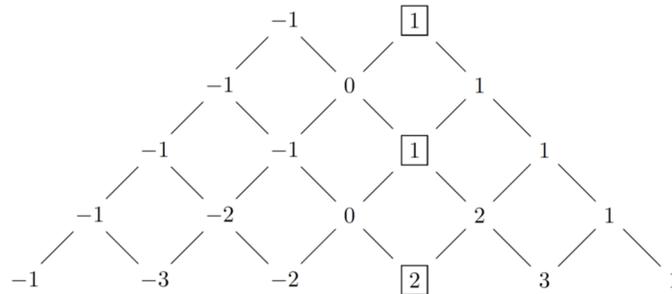


Figure 3. An extension of the Catalan triangle.

Using the entries of Pascal's triangle, we now have a formula for the Catalan numbers. We need only consider how the entries overlap in row $2K$.

Formula for the Catalan numbers. We have $C_K = C(2K, K) - C(2K, K + 1)$.

Exercise. Expand this formula into factorials to obtain $C_K = C(2K, K)/(K + 1)$.

Since C_K is always an integer, it follows that $C(2K, K)$ is always divisible by $K + 1$.

Exercise. Verify the first seven Catalan numbers using either formula. Find the next three values.

Exercise. Use the binomial theorem to show that C_K is the coefficient of x^K in $(1 - x)(1 + x)^{2K}$. Explain the connection to the last exercise in Part 2 and the array above. (Hint: What are the coefficients of x^K and x^{K-1} in $(1 + x)^{2K}$?)

Exercise. Find the general formula for all the entries in the Catalan triangle.

There's still more we can do with Catalan numbers without advanced theory. Let's return to the observation that many loops on the nonnegative integers contain similar smaller loops. If a ballot sequence represents such a loop, it is straightforward to track all returns to zero. Replace each R with 1 and each L with -1. Then the position at time N is the sum of the first N entries, so a return occurs when such a sum equals 0.

For example, ignoring the starting zero, the corresponding position sequences for $K = 3$ are

$$123210, 121210, 101210, 121010, \text{ and } 101010.$$

Exercise. Write the position sequences for the fourteen ballot sequences for $K = 4$.

Given a position sequence, we can readily reconstruct the ballot sequence. Thus, ballot sequences are equivalent to position sequences of length $2K$ that begin with 1, end with 0, contain only nonnegative entries, and have the property that each entry has difference 1 or -1 from the preceding entry. It will be useful to describe the corresponding loops with both types of sequences at the same time.

Our goal is to obtain a formula that counts ballot sequences by tracking the first return to zero. The loops that return to zero only once require special attention. In this case, the ballot sequence must have the form $RR\dots LL$; an occurrence of RL at the beginning or end requires another return to zero. If we remove the outermost letters, the effect is to subtract 1 from each nonzero position and remove the outermost zeros. The new word corresponds to a ballot sequence with length $2K - 2$, possibly representing a loop that returns to zero more than once. For instance, $RRLRLL$ yields $RLRL$ as follows:

$$121210 \rightarrow 010100 \rightarrow 1010$$

Exercise. Verify these assertions when $K = 4$. Describe the effect of removing the outermost letters on the corresponding loops and Dyck paths.

The above process is reversible, so there are C_{K-1} ballot sequences of this form. All other ballot sequences correspond to loops that start with a **proper** loop of length $2N$ and then finish with another loop of length $2(K - N)$. By the Matching Rule, there are $C_{N-1} \times C_{K-N}$ loops whose first return to zero is at time $2N$. Because $C_0 = 1$, summing over these terms gives

Segner's recurrence. For $K \geq 0$,

$$C_K = C_0 \times C_{K-1} + C_1 \times C_{K-2} + \dots + C_{K-1} \times C_0.$$

This formula is the convolution product of Part 2, and we have another way to generate Catalan numbers. Start with the 1-tuple (1), and then

1. Apply the convolution product, denoted by $*$, to the tuple with itself,
2. Append the value from step 1 to the tuple, and
3. Repeat steps 1 and 2 ad infinitum.

For example,

- $(1) * (1) = 1 \times 1 = 1 \rightarrow (1, 1)$
- $(1, 1) * (1, 1) = 1 \times 1 + 1 \times 1 = 2 \rightarrow (1, 1, 2)$
- $(1, 1, 2) * (1, 1, 1) = 1 \times 2 + 1 \times 1 + 2 \times 1 = 5 \rightarrow (1, 1, 2, 5)$, and so on.

Exercise. Calculate the next two Catalan numbers this way.

For a final exercise, we calculate the generating function for the Catalan numbers. Generating functions give a powerful, concise language for expressing combinatorial ideas, especially when

joined with techniques from calculus. For example, with N fixed, the generating function for the binomial coefficients $C(N, K)$ is the familiar polynomial

$$C(N, 0) + C(N, 1)x + C(N, 2)x^2 + \dots + C(N, N)x^N = (1 + x)^N.$$

Exercise. Find the generating functions for the entries of the $2N$ -th rows of the Catalan triangle and its extension illustrated in Figure 3.

For the Catalan numbers, we will need all nonnegative powers of x . We are working purely formally here; one could achieve the same outcome using the infinite sequence C_0, C_1, C_2, \dots

Exercise. Let $F(x) = C_0 + C_1x + C_2x^2 + C_3x^3 + \dots = 1 + x + 2x^2 + 5x^3 + \dots$

1. Use Segner's recurrence to show that $F(x) = 1 + xF(x)^2$.
2. Solve for $F(x)$ by applying the quadratic formula to $xF(x)^2 - F(x) + 1 = 0$.
3. Find the domain of both solutions.
4. What happens to each solution near $x = 0$? Either use technology to graph both solutions, or rationalize the numerator, cancel factors, and evaluate at $x = 0$.

Hint: To help organize the computation of $F(x)^2$ in part 1, you can try multiplying term-wise. Each power of x in the resulting sum requires only finitely many terms.

$$\begin{array}{r}
 1 + x + 2x^2 + 5x^3 + \dots \\
 \times 1 + x + 2x^2 + 5x^3 + \dots \\
 \hline
 1 + x + 2x^2 + 5x^3 + \dots \\
 \quad x + x^2 + 2x^3 + \dots \\
 \quad \quad 2x^2 + 2x^3 + \dots \\
 \quad \quad \quad 5x^3 + \dots \\
 \quad \quad \quad \dots
 \end{array}$$

The preceding discussion barely scratches the surface of the numerous applications of the Catalan numbers. The Catalan numbers are entry A000108 in the On-Line Encyclopedia of Integer Sequences. As noted there, "this is probably the longest entry in the OEIS, and rightly so."

Richard Stanley's book, *Catalan Numbers*, lists over 200 mathematical objects counted by Catalan numbers, starting with Euler's work on triangulations of regular polygons. Segner's recurrence distills the essential Catalan idea to its purest form; when the Catalan numbers appear, we expect to find new objects built from old objects by considering all ways in which complementary parts glue together – fractal in spirit, but with only finitely many steps.

Exercise. Search the internet for a convenient source on Catalan numbers with many examples, preferably with pictures. Identify how Segner's recurrence applies in each example.

We conclude this part of our journey with a joke:

Exercise. What are una, dues, tres, and quatre?

Get it?



The Needell in the Haystack³

Seeing the Data: Classical and Modern Data Visualization Methods
by Anna Ma | edited by Jennifer Sidney

In data science, we don't just leave data in its raw format. In fact, we collect data so that we can process it further! Why not leave the data untouched? This would be the best way to represent the data set, as it is the most faithful to the information we are trying to measure. However, humans are not computers, so it's not easy for us to draw hypotheses or understand the meaning of a vast jumble of numbers. For example, Figure 1 shows just a small snippet (less than one percent of one percent) of a data set. Looking at these numbers, what can you say about the data set? Maybe we can say that the numbers are all less than 1; other than that, there isn't much to be said – and this is just looking at a small fraction of this huge data set! How would you go about making some initial inferences, or even know how to start to think about this data?

0.0157	0.0157	0.0157	0.0157	0.0157	0.0157	0.0157
0.0196	0.0196	0.0196	0.0196	0.0196	0.0196	0.0196
0.0157	0.0157	0.0157	0.0157	0.0157	0.0157	0.0157
0.0157	0.0157	0.0157	0.0157	0.0157	0.0157	0.0157
0.0118	0.0118	0.0118	0.0118	0.0118	0.0118	0.0135
0.0275	0.0275	0.0275	0.0275	0.0275	0.0275	0.0721
0.0157	0.0157	0.0157	0.0157	0.0157	0.0181	0.1993
0.0157	0.0157	0.0157	0.0157	0.0152	0.1676	0.3738
0.0196	0.0196	0.0196	0.0196	0.0703	0.3054	0.4005
0.0157	0.0157	0.0157	0.0157	0.0181	0.1081	0.1708
0.0157	0.0157	0.0157	0.0157	0.0157	0.0157	0.0157
0.0118	0.0118	0.0118	0.0118	0.0118	0.0118	0.0118
0.0196	0.0196	0.0196	0.0196	0.0196	0.0196	0.0196

Figure 1. A very small snippet of a raw data set where the full data set has 1440 rows and 1024 columns of numbers.

Good data visualization allows us to more easily see and interpret the

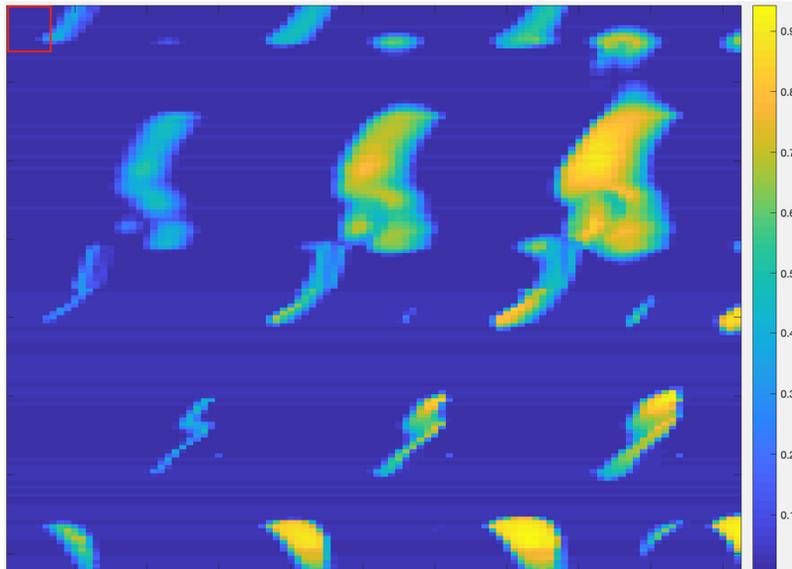


Figure 2. A heat map visualization of roughly 1% of a data set. The red box indicates the visualization of the data from Figure 1.

data, rather than sitting and staring at a big set of numbers. In Figure 2, we have a visual representation of a data set where the red square in the upper left-hand corner indicates the region displayed in Figure 1. The type of visualization shown in Figure 2 is known as a **heat map**. A heat map is a visual representation of data obtained by assigning a color to each number based on the scaling of the value. The color bar on the right tells us how to read the heat map: dark colors indicate that the value is close to 0, and light colors indicate the value is closer to 1. So what

³ This content is supported in part by a grant from MathWorks. Anna Ma is a Visiting Assistant Professor at the University of California Irvine.

observations can be made about this data? In addition to most of the values being smaller than 1, we might notice that most of the values are close to 0 and there seem to be some patterns or repetition in the data. Note that none of these observations would be obvious from Figure 1!

In this article, we will discuss different ways to visualize data. Before reading the article, take a moment to look at Figures 1 and 2. What can you garner from the visualizations? Which ones are easier to interpret? Are some visualizations more appealing, interesting, or revealing than others? (Note that Figure 1 is technically a “visualization” of the data, albeit a literal one.)

Context Matters

Throughout this article, we will represent data sets as a row by column table, or matrix. Each row will be referred to as a data point and each column as a feature. In trying to understand Figure 2, or any visualization for that matter, it’s important to keep in mind the context of the visualization. You might be wondering what the context is for our first example. What do the numbers represent? Figure 1 and Figure 2 are from the Columbia University Image Library (COIL) data set. In this data set, we have a 1440×1024 matrix where each row represents an image and each column an “image feature” (the intensity of an image at a particular pixel location).

Another data set we will consider is the “cities” data set. This data set is a 329 by 9 matrix where each row represents a city, each column represents a category that impacts quality of life, and each value is a scalar associated with a specific category in a particular city. For example, the entry in the 11th row and 2nd column of the matrix is 16047, meaning that the city corresponding to the 11th row (Santa Ana, CA) has a “rating” of 16047 in the housing category. The categories include climate, housing, health, crime, transportation, education, arts, recreation, and economics.

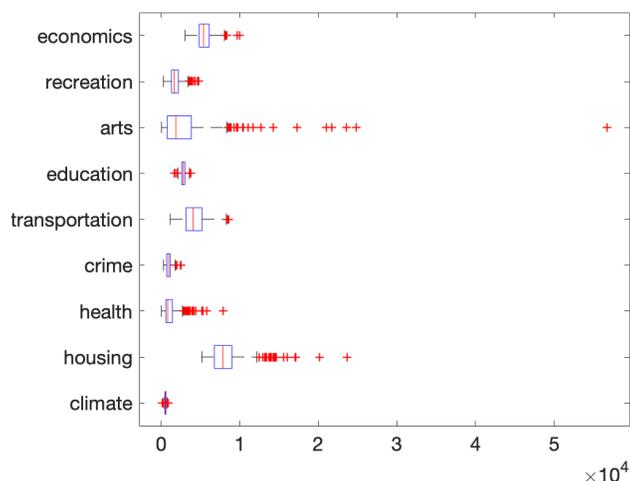


Figure 3. A box-whisker plot for each column of the cities data set.

Figure 3 shows a **box-whisker** plot of the cities data set. A box-whisker plot reveals the adjusted minimum (left whisker), 25th percentile (left edge of box), median (line in the box), 75th percentile (right edge of box), adjusted maximum (right whisker), and outliers (red +’s) for a feature. The “adjusted” maximum and minimum are the maximum and minimum after removing the outliers. In Figure 3, each row corresponds to a column in the cities data set and has its own box-whisker plot, as indicated by the labels on the left-hand side. Notice that there is no label on the horizontal axis in this figure. This is because of a very important fact: each column in the cities data set is at a different **scale** or **unit**. For example, the housing rating data is given in terms of dollars, the crime rating is given in terms of the number of reported crimes per 100,000 persons, and so forth. Thus, in this case, it might make more sense to visualize each feature one at a time.

Classical Data Visualization

There are many ways one can go about visualizing data. So far, we have seen two such examples in the heat map and box-whisker plot. In this section, we will show more examples of different ways to visualize data, including pitfalls that can lead to misleading visualizations.

Figure 4 shows four visualization techniques for the exact same data – the housing rating feature of the cities data set – using a **histogram** (top left), a **scatter plot** (top right), a **line plot** (bottom left), and a **violin plot** (bottom right).

The scatter plot shows the value of every city-housing rating pairing in the cities data set. The horizontal axis is the city index (e.g., city 11 indicates Santa Ana, CA, according to the row labels in the data). The vertical axis tells us the value of the housing rating for a given city. In other words, this is just a visual representation that preserves each data point. We can see that many of the data points fall below 10^4 and that there might be some outliers, but not much else is evident.

One way to garner more information from a visualization is to organize the data in some way. For example, we can group together sets of values into bins and count the number of data points that fall into a particular bin, making a histogram. The histogram is a visualization of data that uses bar heights to indicate the number of data points that fall into a particular range. For example, based off of the histogram in Figure 3, we can see that approximately 290 of the 339 data points fall below 10^4 .

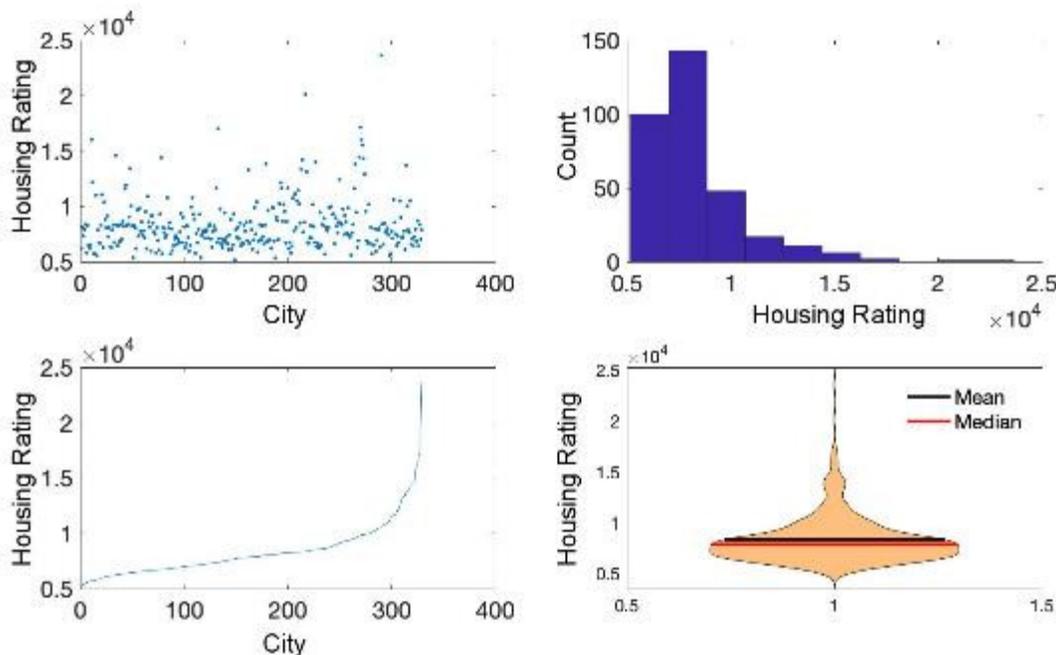


Figure 4. Visualizations of the same housing rating feature from the cities data set: (top left) scatter plot, (top right) histogram, (bottom left) line plot, and (bottom right) violin plot.

We must take care in how we go about organizing the data. One way to organize the data set is to re-sort the values from smallest to largest. This creates a nicer-looking visualization as shown in the line plot in Figure 4, but it could be misleading to the reader. It looks like there is a

relationship between the city index and the housing rating, but this is actually artificially inserted into the data when we sort the housing ratings from smallest to largest.

The violin plot in the bottom right subplot of Figure 4 is very similar to the box-whisker plot shown previously. In fact, if you compare the two, you should see that the median values are pretty close to each other. The main difference between a violin and a box-whisker plot is that the violin plot includes a smoothed version of the histogram you see in the top right plot. In particular, if you draw a curve that closely estimates the histogram, rotate it counterclockwise 90 degrees, and reflect it, we get the violin plot! So, it's just a different way to represent the histogram and box-whisker plots.

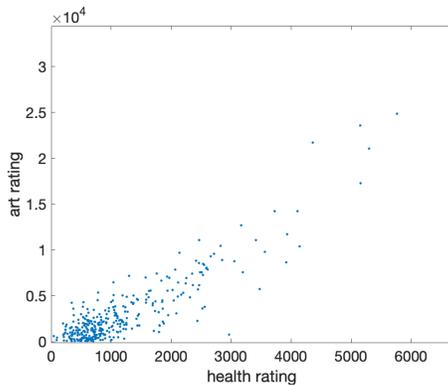


Figure 5. A scatter plot relating the art rating and the health rating of each city in the cities data set.

rating of each city in the cities data set. What does this figure suggest to you about these features? Perhaps we may infer that there is a positive relationship between the art and health ratings. This interesting behavior, revealed to us through a simple scatter plot, can be investigated further at a later stage of data analysis. As another example, Figure 6 presents a three-dimensional histogram that groups cities that are within a specific health rating range and a specific art rating range.

Modern Visualization Techniques

If we don't want to restrict ourselves to two or three dimensions, the problem of data visualization becomes more interesting. Because humans visually can interpret a small number of dimensions, we have to process the data further using data embedding techniques or **dimensionality reduction** in order to simplify the data in some way.

In dimensionality reduction, the goal is to take high dimensional data, for example, a vector $x \in \mathbb{R}^n$, and project or embed it into a lower dimension $\hat{x} \in \mathbb{R}^d$ where $d < n$. We've actually already done some dimensionality reduction in this article! In the cities data set, instead of thinking about the data in dimension $n = 9$, we considered data where $d = 1$ or $d = 2$. In this

Using more features allows us to investigate relationships between multiple features through visualizations. Figure 3 is one way to visualize multiple features at once; however, this could also be misleading, particularly since each feature is in a different scale. In this section, we will demonstrate different ways to visualize higher-dimensional data.

One way to visualize multiple dimensions at once is to use the plotting methods in Figure 3. For example, Figure 5 shows a scatter plot comparing the art and health

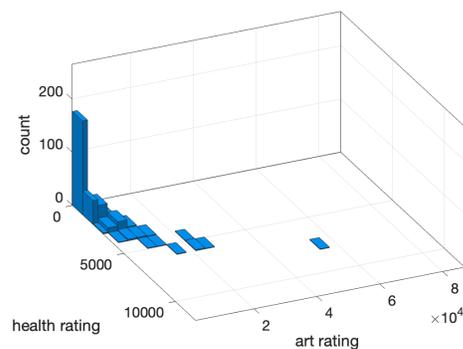


Figure 6. A histogram plot relating the art rating and the health rating of each city in the cities data set.

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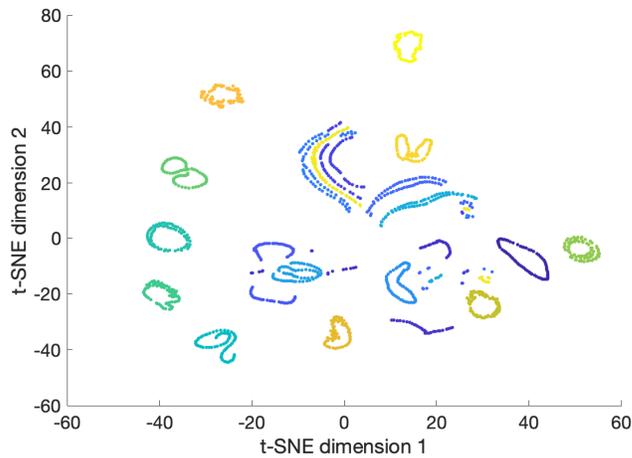


Figure 7. The t-SNE method applied to the COIL data set with label information.

same category are the same color). What do you notice about this visualization? One possible observation is that groups of the data points form rings, and a natural question to ask is whether these rings are meaningful. Again, this is something that can be further investigated at a later stage of the analysis. (Hint: Figure 8 shows 5 images from the one of image categories.)

Data visualization is an important tool used in the exploratory phase of data analysis. Whether it's simple techniques such as histograms, box-whisker plots, or

scatter plots, or more intricate techniques such as t-SNE, the value of data visualization cannot be stressed enough. Data visualization is still a growing field with new methods coming out every year! How would you better present any of the visualizations in this article? Try coming up with your own visualizations by accessing the data sets for COIL and cities in the references below [1, 2].

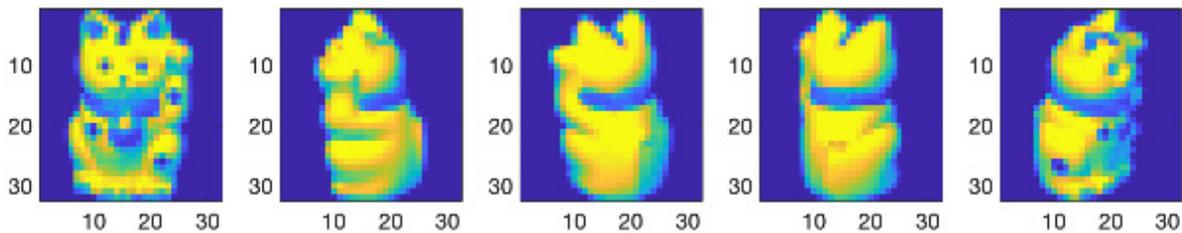


Figure 8. Images from the COIL data set.

References

[1] R. A. Becker, L. Denby, R. McGill, and A. R. Wilks, Analysis of data from the places rated almanac, *The American Statistician*, **41**(3) (1987), 169-186.

[2] [S. A. Nene, S. K. Nayar, and H. Murase, Columbia University image library \(coil-20\), Technical Report CUCS-005-96, 1996.](#)

[3] L. van der Maaten and G. Hinton, Visualizing data using t-SNE, *Journal of Machine Learning Research*, **9**(11), 2008.

Summer Fun!

The best way to learn math is to do math. Here are the 2022 Summer Fun problem sets.

We invite all members and subscribers to send any questions and solutions to us at girlsangle@gmail.com. We'll give you feedback and might put your solutions in the Bulletin!



The goal may be the lake, but who knows what wonders you'll discover along the way?

detours. It's like hiking up a mountain. Getting to the top rewards one with a spectacular view, but during the journey, there's a lot to see and experience. So here's a meta-problem for those of you who feel frustrated when doing these problems: see if you can dissolve that frustration and replace it with a relaxed, optimistic sense of adventure!

This is Summer Fun, not Summer Torture!

In the August issue, we will provide some solutions. You could wait until the August issue to see the answers, but you will learn a lot more if you try to solve these problems on your own.

Some problems are very challenging and could take several weeks to solve, so please don't approach these problems with the idea that you must solve them all. Our main goal is to give you some interesting things to think about.

If you get stuck, try to formulate a related question that you can see a way to actively explore to get your mind moving and your pencil writing. If you don't understand a question, email us.

If you're used to solving problems fast, it can be frustrating to work on problems that take weeks to solve. Try to enjoy the journey and don't be afraid to follow

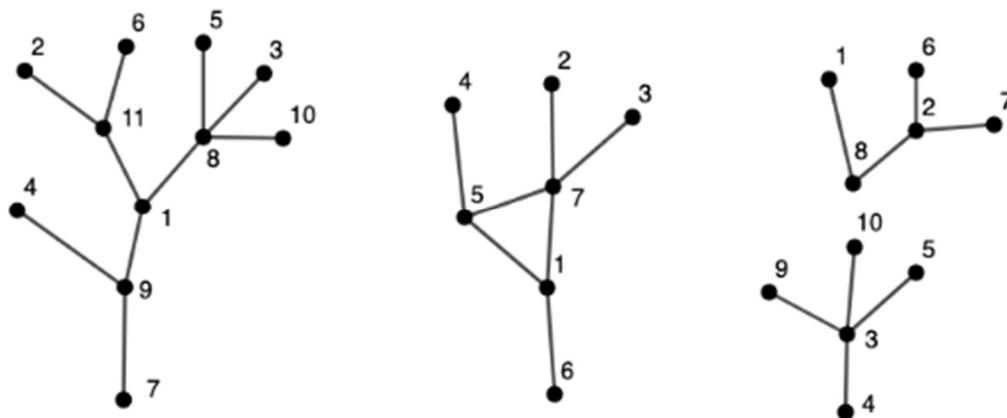
Summer Fun!

Counting Trees and Parking Cars

by Laura Pierson

What are Trees?

A **labeled tree** on vertices labeled $1, 2, 3, \dots, n$ is a way of connecting some of the vertices with edges such that there is exactly one path between any two vertices (so the graph is connected with no cycles). For instance, in the picture below, the first graph is a tree, the second is not a tree because $1, 5,$ and 7 form a cycle, and the third is not a tree because the top part is not connected to the bottom part.



It doesn't matter how the vertices are positioned, only which ones are connected to each other. Two vertices which share an edge are **neighbors**, and a **leaf** is a vertex with only one neighbor. In the tree above, $2, 3, 4, 5, 6, 7,$ and 10 are leaves, while $1, 8, 9,$ and 11 are not.

The Number of Trees

Cayley's tree formula says that the number of labeled trees on n vertices is n^{n-2} . First, let's verify the formula for a few small cases:

1. For $n = 3$, there are $3^1 = 3$ trees. List the 3 trees on 3 vertices.
2. For $n = 4$, there are $4^2 = 16$ trees. List the 16 trees on 4 vertices.
3. If you're feeling up for it, list all the trees on 5 vertices. How many are there?
4. For a challenge, instead of reading on, try to prove Cayley's tree formula.

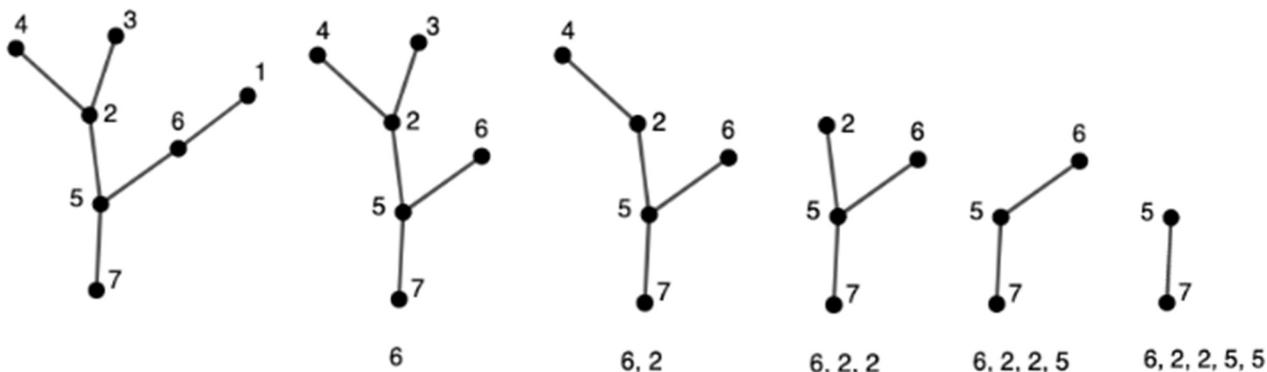
Summer Fun!

Prüfer Sequences

We can prove Cayley's tree formula using **Prüfer sequences**. The idea is to associate each tree to a sequence of $n - 2$ numbers, each between 1 and n , constructed as follows:

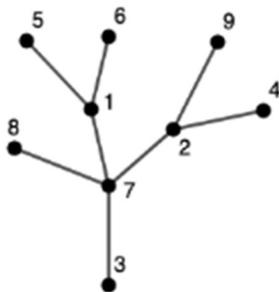
1. Remove the smallest leaf.
2. Write the neighbor of the removed leaf as the next term of the sequence.
3. Repeat until there are only two vertices remaining.

An example of constructing a Prüfer sequence from a tree is shown below. The resulting sequence is 6, 2, 2, 5, 5.



5. Explain why the number of possible Prüfer sequences (given n) is n^{n-2} .

6. Find the Prüfer sequence for the tree below:



7. Find the tree (on 8 vertices) for the Prüfer sequence 5, 3, 3, 4, 7, 2. Explain the process for reconstructing a tree from its Prüfer sequence.

8. Explain why Prüfer sequences give a proof of Cayley's tree formula.

Summer Fun!

Parking Functions

Now let's think about a seemingly unrelated problem! Suppose we have n cars labeled $1, 2, 3, \dots, n$ who are trying to park in n parking spaces, also labeled $1, 2, 3, \dots, n$, and each car has a preferred parking space. The cars will drive to the parking lot in order. On each car's turn, it will first drive to its preferred parking space and see if it is available. If its preferred spot is available, it will take that spot, and otherwise it will keep driving and take the next available spot. The cars cannot go backwards, so if no spots after a car's preferred spot are currently available, the car will leave the parking lot and not get to park. We say a sequence of preferences is a **parking function** if all cars do get to park.

For instance, with 3 cars, $(1, 1, 1)$ is a parking function. Car 1 will park in its preferred space (space 1). Car 2 will see that its preferred space is taken and then take the next available space, space 2. Car 3 will see that its preferred space is taken, so it will keep driving. Since space 2 is also taken, it will take the next available space, space 3. All cars park, so $(1, 1, 1)$ is a parking function.

However, $(3, 3, 3)$ is not a parking function. Car 1 will take its preferred space, space 3. Then car 2 will drive to its preferred space, see that it is taken, and not get to park because it can't drive backwards to the two available spots. Thus, $(3, 3, 3)$ is not a parking function.

9. Find the 3 parking functions for $n = 2$.

10. Find the 16 parking functions for $n = 3$.

11. Now suppose we instead have n cars on a circular track with $n + 1$ parking spaces. In this case, instead of leaving the parking lot once it gets to the last space, a car will just keep driving around, so now all cars always get to park, but there will be one space left unoccupied since there are $n + 1$ spaces and only n cars.

A. Show that the total number of possible preference functions for the cars is $(n + 1)^n$.

B. Explain why each of the $n + 1$ parking spaces is equally likely to be left open.

C. Explain why we get a parking function if and only if space $n + 1$ is left open.

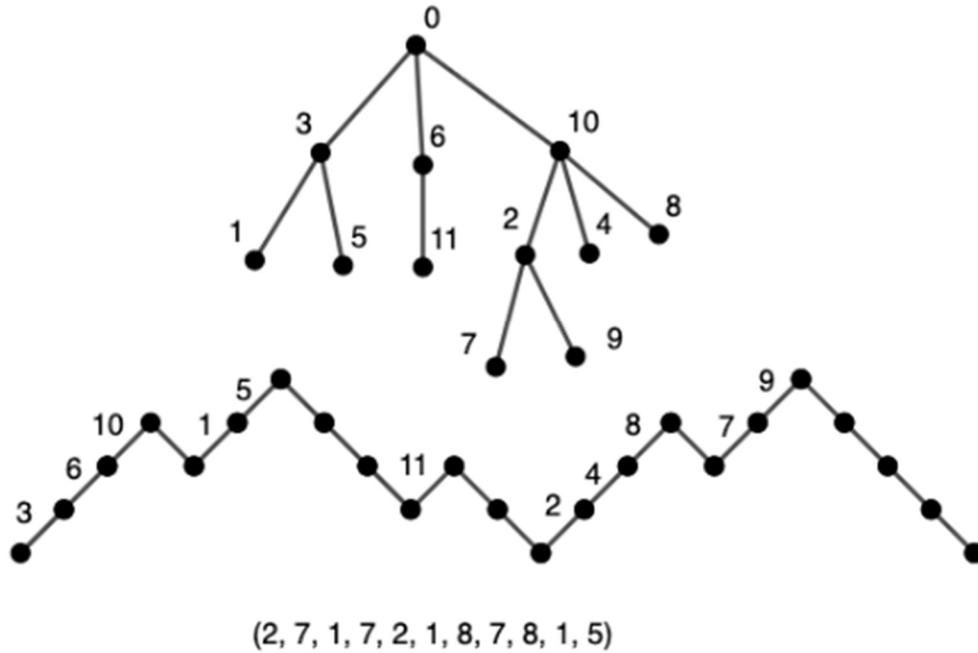
D. Use parts A and B to show that the number of parking functions is $(n + 1)^{n-1}$.

Trees, Paths, and Parking Functions

As we saw above, the number of parking functions for n cars is (surprisingly!) equal to the number of labeled trees with $n + 1$ vertices. We'll now see why that is!



Summer Fun!



In the picture above, we have three different types of objects (with $n = 11$):

1. A tree with $n + 1$ vertices, which we've labeled $0, 1, 2, \dots, n - 1, n$.
2. A **labeled Dyck path**¹ of length $2n$, i.e. a “mountain range” path consisting of n up steps and n down steps that never goes below the horizon, such that each sequence of up steps is labeled with numbers that increase from bottom to top.
3. A parking function for n cars.

12. There is a one-to-one correspondence between these three objects, which shows that there is the same number of each!

A. Using the example above, explain how to go between a labeled tree and a labeled Dyck path.

B. Again using the example above, explain how to go between a labeled Dyck path and a parking function. (*Hint:* In the parking function above, list all the cars who prefer parking space 1. Then list the cars that prefer parking space 2, and so on. What do you notice?)

C. Use parts A and B to explain why the number of trees equals the number of parking functions!

¹For more on unlabeled Dyck paths, see Robert Donley's article on page [INSERT PAGE].

Summer Fun!

Making OJ – The Hard Way!

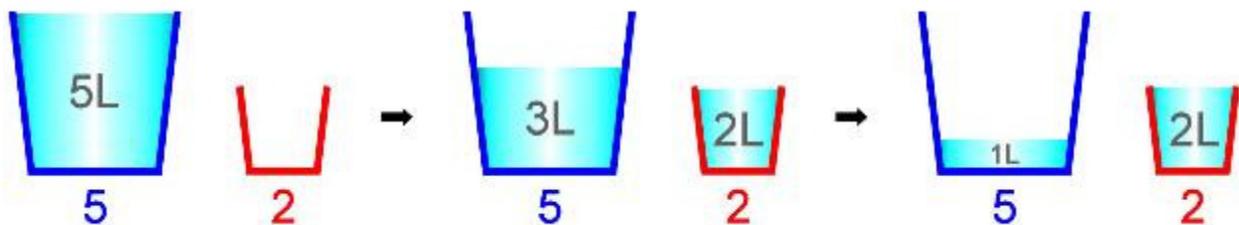
by Mandy Cheung

Let's get two pots and stay next to a water faucet!

Imagine one of pots can hold exactly 2 liters and other pot can hold exactly 5 liters. (These pots are not gradated in any way, so you cannot tell by looking how much water is in a pot, except if it is empty or completely filled.)

1. If you are not that thirsty and just want 1 liter of water, can you get 1 liter of water by using just these two pots?

(Spoiler alert!) Yes, you can! We can first fully fill the 5-liter pot. Then we use the water from the 5-liter pot to fill the 2-liter pot twice. In the end, there will be 1 liter of water left in the 5-liter pot!



Let's try another combination.

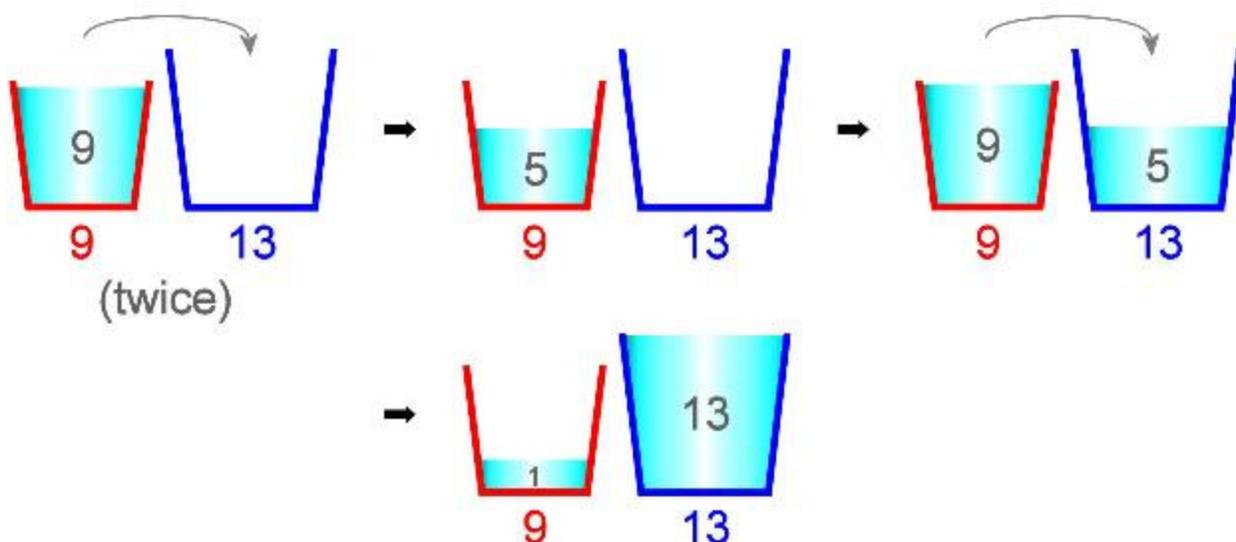
2. Suppose we're now given a 9-liter pot and a 13-liter pot. Can we still get 1 liter of water?

Sure, we can!

(Spoiler alert!) Here's one way: We first fill the 9-liter pot and use the water in it to fill the 13-liter pot. Although someone who didn't see how we did it would not know that there are exactly 9 liters of water in the 13-liter pot, we know because we poured all the water in the filled 9-liter pot into the 13-liter pot. Now, refill the 9-liter pot and then pour water from the 9-liter pot into the 13-liter pot until the 13-liter pot is filled up with water. Because the 13-liter pot already had 9 liters of water in it, there's only enough room for 4 more liters of water in it, and so the 9-liter pot will still contain 5 liters of water. Next, we empty the 13-L pot and then pour the 5 liters of water in the 9-liter pot into the 13-liter pot. We then again fill the 9-liter pot with water and pour water from the 9-liter pot into the 13-liter pot until the 13-liter pot is filled. When this is done, there will be 1 liter of water left in the 9-liter pot!



Summer Fun!



Hm. Can we always get 1 liter of water from two random pots?

3. Let's say this time we have a 2-liter and a 4-liter pot. Show that the smallest, nonzero, amount of water that can be obtained with these two pots is 2 liters.

Now we want to make orange juice. But not just any orange juice. We want to make it to a concentration of our preference. Can we dilute 100% juice to whatever portion we enjoy by using two cups with known measurement, and numerous cups of unknown sizes?

4. We return to the setup in Problem 1 where we had a 2-liter and a 5-liter cup, but we add numerous cups of unknown large sizes. (By "numerous," we mean that you can use as many of these cups as you wish, and by "unknown large sizes," we mean that you can, in practice, fill them with any amount of liquid without overflowing, but you cannot tell how much liquid is inside by observation.) You have taps that provide water and 100% orange juice, fresh squeezed. Can you produce a liter of 60% orange juice? What about a liter of 50% orange juice? What about a liter of 25% orange juice?

5. Suppose you have an x -liter and a y -liter cup, where x and y are positive integers, as well as numerous cups of unknown large sizes. There are two taps that provide an unlimited supply of water and 100% orange juice. What concentrations of orange juice can you produce?

Be Irrational!

6. Now suppose you have access to a tap that produces an unlimited supply of water, a 1-liter cup, a $\sqrt{2}$ -liter cup, and numerous cups of unknown large sizes. It's not possible to produce exactly $1/2$ liter of water, but you can get awfully close. Can you think of a way to produce $1/2$ liter of water within an error of no more than $1/100$?

Summer Fun!

Fermi Questions

by AnaMaria Perez and Josh Josephy-Zack

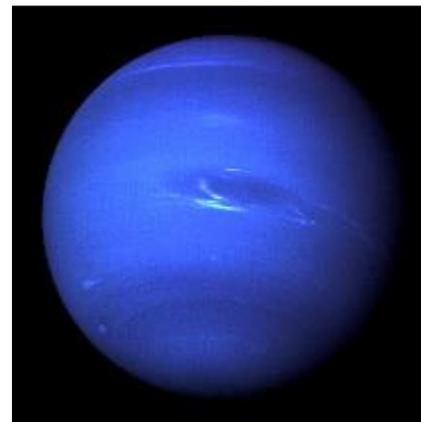
Fermi questions are questions that are impractical, or even impossible, to answer exactly, so you have to give your best estimate. Many require facts about the world in general and so are not strictly speaking, purely math questions. However, they can be a lot of fun, and that's what Summer Fun is all about.

1. How many circles are on the cover of this issue? (Yes, in your estimate, include circles that are covered by the banner of article titles.)
2. What is the product of the number of Starbucks locations in the US and the number of spectators who attended the 2021 Tokyo Olympics?
3. What fraction of the average American female lifespan is the blink of an eye?
4. If you unraveled all the wool inside of a baseball, how long would it reach (in terms of the height of Fenway Park's Green Monster wall)?
5. How many standard 3×3 Rubik's Cubes can an Olympic-sized swimming pool hold?
6. How much energy (in yottawatt-hours) does a Boeing 747 use when flying from London to New York? (One yottawatt-hour is 10^{24} watt-hours.)
7. On average, how many individual playing cards does a small casino on the strip in Las Vegas use every year?
8. Assuming he never gets tired, how many lightyears could Usain Bolt run at top sprinting speed in one week?
9. How many pinheads would it take to fit the entire farmland of Kansas?
10. What is the product of the number of cases of the common cold in the United States each year and the number of letters in the Constitution of the United States?
11. Assuming unlimited resources and a constant work pace, how long would it take for an average Krispy Kreme donut shop to bake enough donuts to stack together (top-to-bottom, not side-to-side) to reach the Moon?
12. How many milligrams of caffeine are in the entire volume of Coca-Cola sold in the United States on an average day?



Summer Fun!

13. Using the power output of the Sun, how long (in millennia) would it take to charge an iPhone 13?
14. If all of the oxygen in Earth's atmosphere were confined to the volume of a Magic 8 Ball (assuming it can all fit), what would the density be in grams per cubic centimeter?
15. How many carbon dioxide molecules would you have to stack end-to-end to reach the top of the world's tallest tree?
16. If all the copies of Harry Potter ever sold were stacked cover-to-cover on a single bookshelf, how long would this bookshelf be (in terms of the average length of an adult earthworm)?
17. How many Amazon delivery vans could you fill with the average annual rainfall in the Amazon rainforest?
18. What fraction is the mass of an atom of titanium to the mass of the planet Neptune, squared?
19. How many pencils would it take to draw a line around the equator of Mercury, assuming the pencils are all continuously sharpened and each used up completely?
20. Assuming current power consumption, how many Twinkies would you need to power New York City for a century? (Here, we're thinking of the amount of energy in a twinkie to be its chemical energy content as a food.)



Neptune, courtesy of NASA

BONUS

For the following questions please provide the tightest range you can. Submit it to this google form, and the tightest correct range will receive a prize! <https://forms.gle/p8ZCey9gyLwyurAK9>

Bonus 1. What is the product of the two problem set creators' favorite numbers?

Bonus 2. We picked two points independently at random inside a unit circle and computed the area of the triangle whose vertices are the two points together with the center of the circle. We did this twice, once for each of the following random distributions. For each distribution, what area did we compute?

A. A uniform distribution over the circular disc.

B. A uniform distribution in polar coordinate space. That is, we found each point by picking r uniformly at random on the interval $[0, 1]$ and θ uniformly at random on the interval $[0, 2\pi]$ and taking the point to be (r, θ) in polar coordinates.

Summer Fun!

Notes from the Club

These notes cover some of what happened at Girls' Angle meets. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

Session 30 - Meet 12 Mentors: Mandy Cheung, Cecilia Esterman, Jenny Kaufmann, May 5, 2022 Bridget Li, Tharini Padmagirisan, Kate Pearce, Laura Pierson, Vievie Romanelli, Sakshi Suman, Jane Wang, Rebecca Whitman, Muskan Yadav, Angelina Zhang

Our last meet of Session 30 saw another First at Girls' Angle. Our traditional end-of-session Math Collaborations have been buffeted by pandemic winds. We cancelled Session 29's event, and were going to cancel Session 30's. However, members Eva Arneman Altea Catanzaro, Saideh Danison, and Viveka Mirkin came to the rescue by creating, completely from scratch, a Math Collaboration for all of us to enjoy!

A typical Math Collaboration requires construction of two to three dozen problems all couched within a momentous story arc. It's quite an undertaking! So a huge round of applause to our Math Collaboration construction team, who worked under the supervision of mentor Rebecca Whitman. Try your hand at a few of their problems:

Decipher this secret code.
The answer is an English sentence.

Ojz hsozct fw Zkktncsga scz cgwz hgoj kgwz, gaykbpagan nkfhgan lzkkewgtj, tzs obcokzs, qkfqwgtj, sap aschjskt.

Find the number of each animal:

- The number of glowing jellyfish is 6 times the number of narwhals.
- All the numbers are even except for the number of sea turtles.
- The number of glowing jellyfish divided by 9 is the amount of blobfish.
- None of the numbers are bigger than 40.
- The number of glowing jellyfish is 15 more than the number of sea turtles.
- All of the numbers are whole numbers.

1. Viviana Elliott has never shown any interest whatsoever in biology.
2. The scientist who majored in neuroscience teaches at MIT.
3. Jeremy is doing his PhD at a school in England.
4. The scientist whose last name is Kleinman is not Sophie.
5. Ms. Foster is studying at Oxford.
6. Neither the scientist surnamed Kleinman nor the one at Tufts study marine botany.
7. If Elizabeth's last name is Flowers, then Viviana studies marine botany.
8. The scientist at Cambridge has loved marine botany since age 5.

	Kleinmen	Foster	Flowers	Elliott	Tufts	MIT	Cambridge	Oxford	Marine Botany	Chemistry	Biology	Neuroscience
Sophie												
Jeremy												
Elizabeth												
Viviana												
Marine Botany												
Chemistry												
Biology												
Neuroscience												
Tufts												
MIT												
Cambridge												
Oxford												

Calendar

Session 30: (all dates in 2022)

January	27	Start of the thirtieth session!
February	3	
	10	
	17	
	24	No meet
March	3	
	10	
	17	
	24	No meet
	31	
April	7	
	14	
	21	No meet
	28	
May	5	

Session 31 (tentative schedule): (all dates in 2022)

September	8	Start of the thirty-first session!
	15	
	22	
	29	
October	6	
	13	
	20	
	27	
November	3	
	10	
	17	
	24	Thanksgiving - No meet
December	1	

Girls' Angle has run over 150 Math Collaborations. Math Collaborations are fun, fully collaborative, math events that can be adapted to a variety of group sizes and skill levels. We now have versions where all can participate remotely. We have now run four such "all-virtual" Math Collaboration. If interested, contact us at girlsangle@gmail.com. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

Girls' Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) _____ (first) _____

Parents/Guardians: _____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: _____ Cell Phone: _____

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

The \$50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- Enclosed is a check for \$50 for a 1-year Girls' Angle Membership.
- I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.



A Math Club for Girls

Girls' Angle Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is \$20/meet for members and \$30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle is based in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, founder and director of the Exploratory
Yaim Cooper, Institute for Advanced Study
Julia Elisenda Grigsby, professor of mathematics, Boston College
Kay Kirkpatrick, associate professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, assistant dean and director teaching & learning, Stanford University
Lauren McGough, postdoctoral fellow, University of Chicago
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, associate professor, University of Utah School of Medicine
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Liz Simon, graduate student, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, associate professor, University of Washington
Karen Willcox, Director, Oden Institute for Computational Engineering and Sciences, UT Austin
Lauren Williams, professor of mathematics, Harvard University

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) _____ (first) _____

Parents/Guardians: _____

Address: _____ Zip Code: _____

Home Phone: _____ Cell Phone: _____ Email: _____

Please fill out the information in this box.

Emergency contact name and number: _____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

(Parent/Guardian Signature) Date: _____

Participant Signature: _____

Members: Please choose one.

- Enclosed is \$216 for one session (12 meets)
- I will pay on a per meet basis at \$20/meet.

Nonmembers: Please choose one.

- I will pay on a per meet basis at \$30/meet.
- I'm including \$50 to become a member, and I have selected an item from the left.

I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

**Girls' Angle: A Math Club for Girls
Liability Waiver**

I, the undersigned parent or guardian of the following minor(s)

_____ ,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: _____ Date: _____

Print name of applicant/parent: _____

Print name(s) of child(ren) in program: _____