

Girls' *Angle* Bulletin

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To Foster and Nurture Girls' Interest in Mathematics



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From the Founder

Suppose you and your friends find yourself stuck on a deserted island for days. As boredom sets in, you decide to liven things up with a game, but you have no games. What can be done? *You* can create a game. Is the game fun? No? Then *you* can change the rules! - Ken Fan, President and Founder

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Girls' Angle: A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

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On the cover: *Seen at Last*. Original photo: Michael Hutchings. Modifications: C. Kenneth Fan. For meaning, see page 26.

An Interview with Tullia Dymarz, Part 3

This is the concluding part of our interview with Professor Dymarz.

Ken: May I offer a characteristic that I think you possess [that helps with doing math]? [Laughter.]

Tullia: Sure.

Ken: You exude this craving for a deeper understanding and an optimism that a deeper understanding exists. Is that something you always had? How do you know that there is more to understand?

Tullia: That's interesting, because when it comes down to it, I don't think I am optimistic! When someone says, "Oh, I think this can be done this way," I'm very pessimistic about it.

Ken: That's funny!

Tullia: Actually, if you're collaborating, I think you need a bit of both. You need someone to be the optimist and someone to be the pessimist. You need the optimist to try outlandish ideas that the other person thinks won't get anywhere, and you need the pessimist to rein someone in a little bit.

It's also very different, I think, working on something that has never been solved. Your level of optimism and pessimism can be very different than your feelings about math. But, yes, you do need a little optimism to even be able to attempt some of these hard problems. So, yes, optimism is a good quality to have, and maybe I need a little more of it!

Ken: What about your sense for where there is interesting math to be had? What guides you on that front?

Tullia: It's hard to say. It's based on what you've already encountered and what you've thought deeply about. And sometimes it comes out of talking to other people, and hearing about what they're curious about, and, oftentimes, trying to combine the two, or realizing that you understand something about the question they're asking.

I think that's how a lot of math builds on itself, and how people find interesting things to work on. Some people can work alone and pursue their own program, but I think a lot of it comes out of talking to others.

Ken: You run a program for high school girls called *Girls Math Night Out*. Please tell us about that program.

Tullia: It was started by a now associate dean, Gloria Marí Beffa. She'd be able to give you more details. I basically took over this program roughly when I arrived here. She had been running it for a while and wanted to move on to other projects. I liked it because it was a noncompetitive math activity, very different from doing math contests.

I also thought it would be cool to get to know some of the high school girls who are interested in math. I didn't go to school in the US, because I grew up in Canada. I knew I was going to be in the States longer, so I wanted to see what their high school experience was like. I also wanted to provide opportunities for our grad students to do some mentoring. And I just thought it was a good program and I enjoy being part of it.

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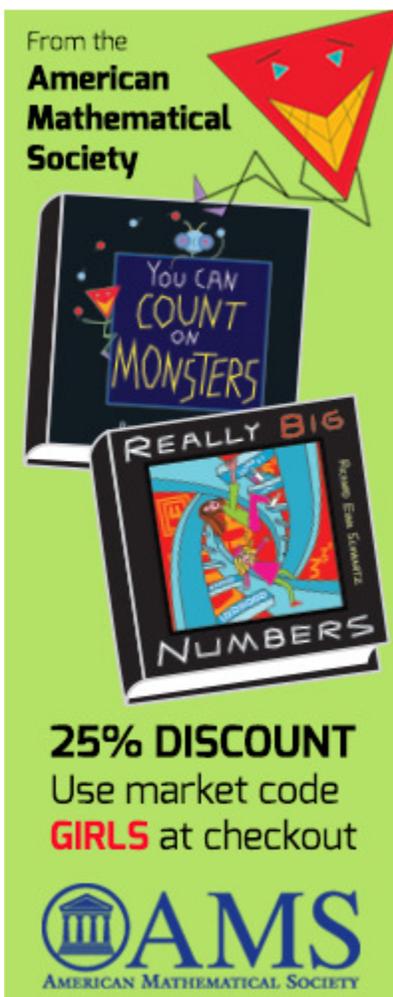
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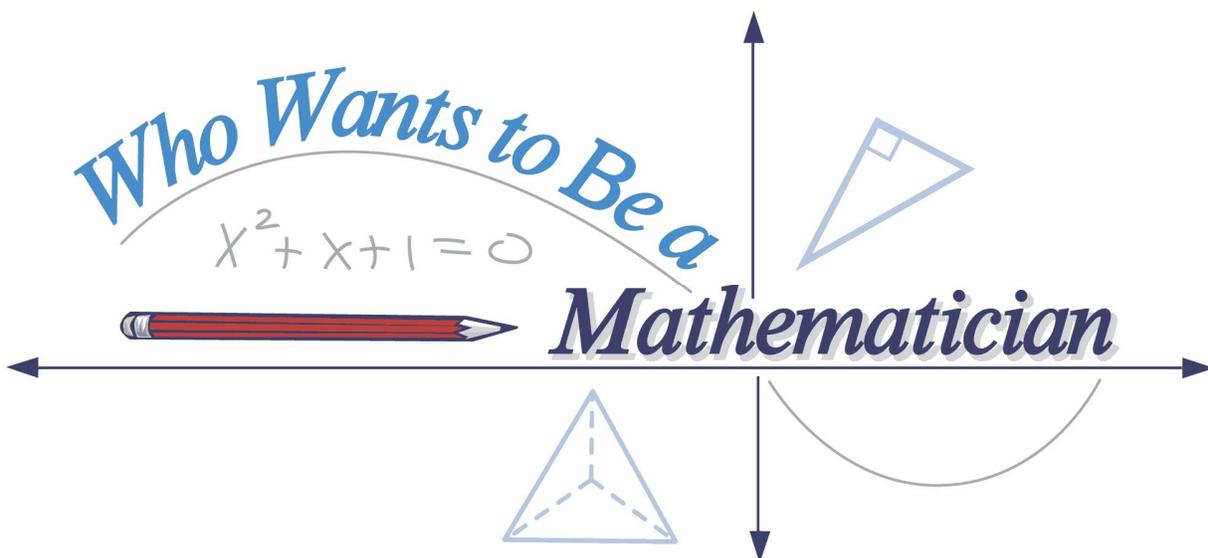
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Ken Fan
President and Founder
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The Needell in the Haystack¹

What Is Data: Representing Information as a Matrix

by Anna Ma | edited by Jennifer Sidney

Have you ever wondered why matrices and vectors are so ubiquitous in data science and machine learning? And what do we mean by *data*; what is it? How can we take information and turn it into an object we can work with mathematically? In this installment, we explore a variety of ways we can represent the world as numbers, and ask interesting questions framed as mathematical problems. This fun exercise is at the core of data science!

Information is all around us. Look around the space you're in now and experience that room through your senses. What do you see? What is the temperature of the room? Can you hear any sounds? Are there any distinct smells? As I'm writing this article, I can tell you the following about my environment: I see, among other things, two cats, a Hydro Flask water bottle, my cell phone, and a coffee table. The temperature is a nice, cozy 68°F (thank you, wonderful Southern California weather!). I hear typing and I can smell the espresso I made this morning. These are all pieces of information about my surroundings.



Figure 1. The cats in my room.

Data as Scalars

To collect this information in the form of **data**, we want to represent the information in a numerical way. For example, let's have x represent the number of pets that are in the room. In my case, $x = 2$. The variable x is **scalar** data, because a scalar is a number and x represents the number of pets in a room. We could also count the number of pets in your room and use subscripts to indicate whose room the data is collected from; assuming you have one pet in your room, we can write $x_{Anna} = 2$ and $x_{you} = 1$, and now we have two pieces of scalar-valued data!

If we ask friends and classmates for the number of pets in their room, it might get a little cumbersome to keep writing everyone's name, so let's instead assign **indices**² to each person. I will be index 1, you will be index 2, and so on. After collecting this information, we may end up with something that looks like this:

$$x_1 = 2, x_2 = 1, x_3 = 0, x_4 = 5, x_5 = 0, x_6 = 1, x_7 = 2.$$

This set of scalar data tells us the following: person 1 (me) has 2 pets in their room, person 2 (you) has 1 pet in their room, person 3 (friend) has 0 pets in their room, person 4 has 5 pets in their room, and so on.

¹ This content is supported in part by a grant from MathWorks. Anna Ma is a Visiting Assistant Professor at the University of California Irvine.

² Plural for index.

Data as Vectors

Instead of writing this as 7 scalar-valued data, we could instead represent these data as a single **vector**. A vector is simply a list of scalars. Now, instead of 7 scalars, the data representing the number of pets in a person's room is represented by a vector:

$$x = [2, 1, 0, 5, 0, 1, 2].$$

A vector is composed of **entries** that only require one **index** to pinpoint. For example, the 4th entry of x is the value 5, or in mathematical notation, $x_4 = 5$. Every vector also has a **dimension**, which is the number of scalars in that vector. In this example, x is a seven-dimensional vector.

Data as Matrices

Now, what kinds of questions can be asked and answered with this data? These are some examples: What's the average number of pets in a person's room? How many people have more than three pets in their room? How many people don't have any pets in their room? Here, it's important to keep in mind what information the data is actually representing. We cannot accurately answer the following questions: How many rooms have cats? How many rooms have more than one type of pet? Beyond that, we certainly can't answer these questions: Does having a cell phone in your room impact your homework grades? Does having more pets mean having a warmer room? Are people working in a room with pets more likely to succeed in school? To answer these other questions, we're going to need more – you guessed it – data!

We have scalar values and we have vectors, which are lists of scalar values. We can also have lists of vectors (which are lists of lists of scalar values). These objects are called **matrices**.³ Here's an example of a matrix: suppose for each person, Person 1 to Person 7, we asked the following collection of questions:

- a. How many pets do you have in your room?
- b. What is the temperature of your room (in degrees Fahrenheit)?
- c. What percentage grade did you get on your last homework assignment?
- d. How many cell phones are in your room?
- e. How many pens/pencils do you have with you?

For each of these questions, the information collected can be represented by a vector, as we did with question (a) and vector x above:

$$\begin{aligned}a &= [2, 1, 0, 5, 0, 1, 2] \\b &= [68, 70, 60, 55, 65, 65, 63] \\c &= [100, 100, 90, 100, 80, 85, 81] \\d &= [1, 1, 0, 0, 0, 2, 1] \\e &= [2, 4, 2, 2, 0, 5, 1]\end{aligned}$$

³ Plural for matrix.

Now we organize these vectors into a matrix by stacking them, one on top of the other, like so:

$$X = \begin{bmatrix} 2 & 1 & 0 & 5 & 0 & 1 & 2 \\ 68 & 70 & 60 & 55 & 65 & 65 & 63 \\ 100 & 100 & 90 & 100 & 80 & 85 & 81 \\ 1 & 1 & 0 & 0 & 0 & 2 & 1 \\ 2 & 4 & 2 & 2 & 0 & 5 & 1 \end{bmatrix}.$$

This is our first **data matrix**: it holds data about surrounds and grades of different people. Matrices have dimensions given by the number of their rows and the number of their columns. By convention, the number of rows is always listed first. Thus, this is a 5 by 7 matrix.⁴ As with vectors, matrices are composed of entries. Entries of a matrix require two indices to pinpoint: a row index and a column index. For example, the entry in the 3rd row and 5th column of X is denoted $X_{3,5}$. Here $X_{3,5} = 80$.

Although matrices contain scalar valued entries, this doesn't mean that we can only represent numerical data. For example, what if we have categorical information – “Is your pet orange, gray, or white?” – text data (tweets, news articles, etc.), or images? In the following examples, we will show how different types of information can be represented by these versatile objects.

Example 1: Numerical Data - Netflix Data Set

Netflix user-movie information can be represented using a matrix. A visual example appears in Figure 2. Here, we have an $m \times n$ matrix where there are m users and n movies. Each row represents a user (left image), each column represents a movie (center image), and for each user-movie combination we have some scalar-valued data (right image). For example, for each user-movie pairing, we could have the rating that the user gave the movie on a 5-star scale. So, if User 5 did not like *Inside Out*, we would see a value 1 in the orange block of our matrix. If we wanted other numerical information, such as the number of times a user watched a movie, this would be a data matrix with the same number of rows and columns but with different entries.

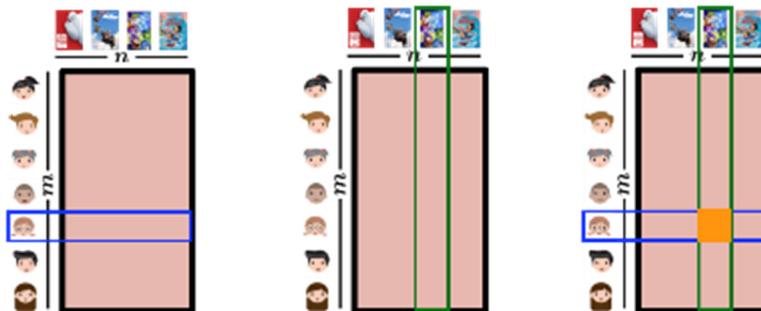


Figure 2. Example of an $m \times n$ data matrix for user-movie data where (Left) rows represent users, (Center) columns represent movies, and (Right) entries are for each user-movie pairing.

⁴ We can also think of vectors and scalars as special types of matrices: a vector is an $n \times 1$ or $1 \times n$ matrix, and a scalar is a 1×1 matrix.

Example 2: Text Data - News Article

If we have text, there are natural ways to represent that data in a matrix. Consider, for example, a set or corpus of news articles from different news outlets. Let X be an $m \times n$ data matrix where m is the number of articles and n is the number of unique words that appear across all articles. In Figure 3, we have a visual example of such a data matrix. Each row represents a different news article from a news outlet, and each column represents different words that appear (or don't appear) in an article. For each entry of the matrix, we can record the number of times that word appears in the article.

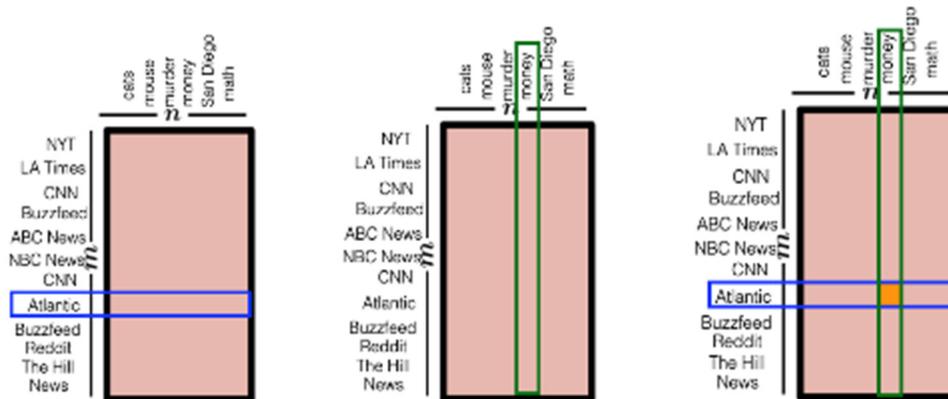


Figure 3. Example of an $m \times n$ data matrix for article-word data where (Left) rows represent news articles, (Center) columns represent words, and (Right) entries are for each article-word pairing.

As a more concrete example, we suppose the following sentence appears in the *Atlantic* article:

“Cats in San Diego can make money chasing birds away from strawberry fields. The math definitely works out for having cats!”

Because the word “Cats” appears twice in the sentence, “San Diego” once, “money” once, and “math” once, and the words “murder” and “mouse” don’t appear, the highlighted row in Figure 3 would be the vector

$$x_8 = [2, 0, 0, 1, 1, 1].$$

Example 3: Image Data - Cats

For our last example, we consider the image of the cats in Figure 1. How can we represent this image as a matrix? Turns out, it was already represented as one! The image in Figure 1 was printed from a digital image of my cats. Digitally, the image is stored as a two-dimensional matrix, where the dimensions are the number of pixels that make up the height and width of the image and the entries are the gray-scale intensity of the image. In other words, Figure 1 represents an $m \times n$ matrix X where m is the height of the image in pixels and n is the width of the image in pixels. The scalar entries of the matrix range from 0 to 255 where 0 means that pixel should be colored black, 255 means the pixel should be colored white, and all the numbers in between are gradient colors between black and white. Figure 4 shows an example of a small part of the image represented as a large matrix.

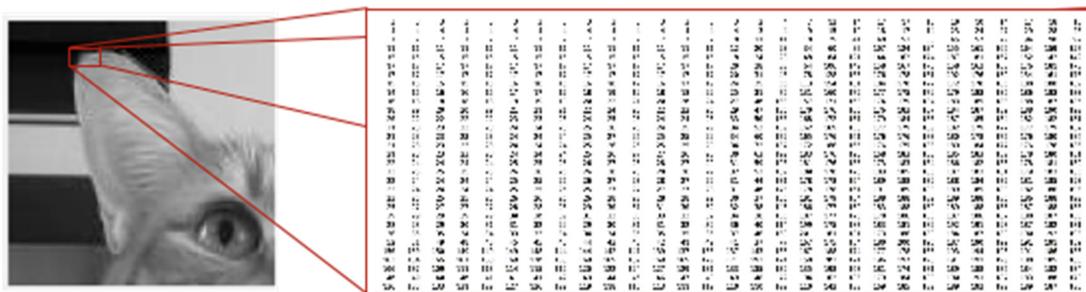


Figure 4. Matrix representation for part of the image presented in Figure 1.

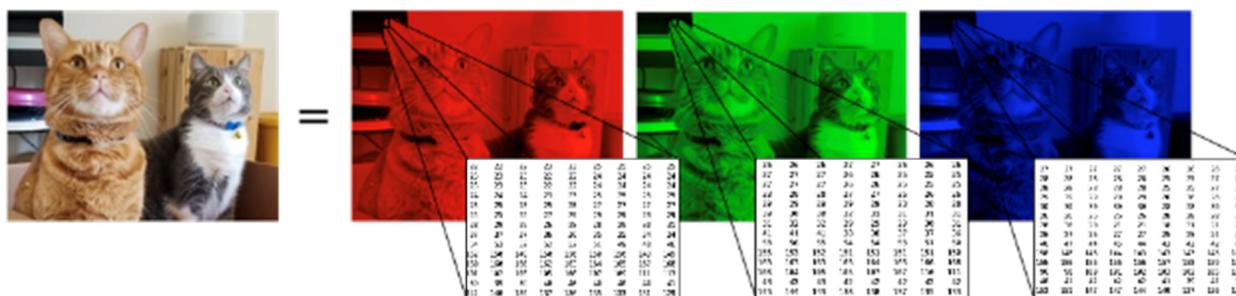


Figure 5. RGB decomposition of the colored version of Figure 1.

Beyond Making the Matrix

A natural question to ask is whether we can go beyond the two-dimensional matrix to other formats for organizing data. This is the answer: you can create whatever format you wish, and there's no limit on the possibilities. One generalization is to use higher-dimensional arrays known as **tensors**. For example, RGB images (short for Red-Green-Blue images) are a set of three matrices in which each matrix represents the intensity of the three different primary colors for a given pixel in an image. See Figure 5 for an example of the RGB decomposition of the colored version for Figure 1. If we stack the colored slices of the image one behind the other, we end up with an $m \times n \times 3$ tensor where each entry of the tensor indicates the intensity of a particular color. For example, the entry indexed by $(30, 40, 3)$ of the tensor indicates how much blue color is in the pixel in position $(30, 40)$. Notice that the orange cat (orange is mainly a combination of the red and green colors) is easier to see in the red and green matrices. Tensors are not simply limited to image data. Can you come up with examples of tensor data with numerical information? What about with text?

Information is all around us. It's through the advent of new and exciting technologies that we have access to this information in the form of data. Data can appear structured as a variety of mathematical objects ranging from vectors to tensors. Now that we have a foundation for taking information and turning it into something we can use in the world of mathematics, we can focus on asking questions that want to use our data to answer. For example, using pictures of cats to "learn" what cats generally look like, can I determine whether a new picture is a picture of a cat? Using historical information about people's surroundings and how successful they were on homework, can I create an optimal environment for success? What type of questions would you be interested in asking, and what kinds of data do you think you would need to answer such questions? Email me at anna.ma@uci.edu if you have a question and data set that you are interested in pursuing!

Shortcuts to Counting

by Robert Donley

edited by Amanda Galtman

In Volume 13, Number 3 of this Bulletin, Esmé Krom and Molly Roughan reported new results that they found connecting certain paths in a triangular network to the Euler numbers. Lattice path counting is a fairly active subject in mathematics; it is the focus of the recurring International Conference on Lattice Path Combinatorics and Applications.

If you have heard about Pascal's triangle, you may know some basic path counting results. We'll get to that at the end, but first let's review some basic counting techniques from combinatorics, which includes the subject of counting things (enumeration).

First, we have the Matching Rule. This is a quick way to count if we make several choices that do not depend on each other; the power of the Matching Rule is that we get the number of outcomes without having to list all of them.

Matching Rule: If our first choice has M outcomes and our second choice has N outcomes regardless of the first choice, then there are $M \times N$ possible outcomes.

For example, if we have three shirts and two pairs of pants, we can form six outfits. To check this, if we label the shirts A, B, C , and the pairs of pants 1 and 2, then the six possible outcomes are $A1, A2, B1, B2, C1$, and $C2$.

Exercise: How many outfits are possible if we have five shirts and four pairs of pants? Check by listing all outcomes.

Exercise: In the previous exercise, what happens if we also have three pairs of socks? Would you want to list all the outcomes by hand?

You can think of the Matching Rule using a rectangular arrangement of outcomes. This arrangement makes the listing of outcomes more efficient and explains why the rule uses multiplication.

1	A1	B1	C1
2	A2	B2	C2
	A	B	C

Exercise: List the outcomes from the first exercise using a rectangle. Building on this list, you can try to list the 60 cases in the second exercise as a 20-by-3 rectangle. Alternatively, do you see a way to do the listing in the second exercise using three dimensions?

Now let's apply the Matching Rule to obtain some fancier counting techniques.

Suppose we have five different math books and we want to arrange them on a shelf. How many ways can we line them up?

Let's note the important points here: order matters and each position counts as a choice. There are five ways to pick the first book in the ordering. How many possibilities are there for the second book? We can't reuse the first book, so four books remain. One thing to note: these four books depend on the outcome of the first choice, but there are always four books to choose from no matter the first choice. We can apply the Matching Rule because the *number* of possible outcomes for the second choice does not depend on the first choice. Hence, we see that there are $5 \times 4 = 20$ ways to line up the first two books. To line up three books, we note that three books remain after the second choice, so we multiply 20 by 3 to get 60. Two books remain for position 4, so there are $60 \times 2 = 120$ ways to line up four books. When we get to the fifth book, we have only one choice, so we conclude that there are $5 \times 4 \times 3 \times 2 \times 1 = 120$ different ways to line up our five math books in a row. Each such arrangement is called a **permutation**. To generalize,

Permutation: The number of ways to order N different objects in a row is

$$N \times (N - 1) \times (N - 2) \times \dots \times 3 \times 2 \times 1.$$

Multiplying the numbers from 1 to N is so common that mathematicians created a shorthand notation for this product called **factorial** and it is written: " $N!$ ". We pronounce $N!$ as " N factorial." As a convention, we define $0!$ to be 1. This convention makes it easier to write certain formulas.

Exercise: Note that $3! = 6$. Verify the number of permutations by listing all orderings of A , B , and C . How many ways are there to write the letters A , B , C , and D in a row? Verify by listing them all.

Exercise: Calculate $N!$ for as many terms as you can by hand. Would you want to list all orderings of the letters A , B , C , D , and E by hand?

Next, let's consider orderings, but suppose that not all items are used. For example, suppose we have 10 books to line up, but our shelf has room for only six. We can actually apply the same reasoning we used for counting permutations, except that we have to stop after we've placed six books. Thus, the number of outcomes is $10 \times 9 \times 8 \times 7 \times 6 \times 5 = 151,200$.

To express such products, we introduce the notation $P(N, K)$ to stand for

$$N(N - 1)(N - 2)\cdots(N - K + 1)$$

Exercise: Show that $P(N, K) = \frac{N!}{(N - K)!}$.

A way to order K objects from a collection of N distinct objects is known as a **partial permutation** or, more precisely, a **K -permutation**.

Another way to think of the formula for $P(N, K)$ is in terms of how we overcount with factorial. For example, we can think of $10!$ as two operations: first, we order any six elements, and then we order the remaining four. Since the number of ways to order the remaining four books does not depend on the choice or order of the first six books, this gives

$$10! = P(10, 6) \times 4!$$

or

$$P(10, 6) = 10!/4!.$$

Exercise: Do our formulas for $P(N, K)$ make sense when $K = 0, 1$, or N ?

Exercise: Suppose we order two elements of $\{A, B, C, D\}$. Calculate $P(4, 2)$ using both formulas for $P(N, K)$, and list each ordering.

Now suppose we wish to choose K books from N books, but the order does not matter. We denote the number of ways to do this by $C(N, K)$ and read this notation as “ N choose K .” As with permutations, there are several ways to derive a formula for $C(N, K)$.

Here’s one way: we can consider ordering six out of 10 books as two independent choices: first we choose six books from the 10, and then we order the six books. Using the Matching Formula,

$$P(10, 6) = C(10, 6) \times 6!$$

or

$$C(10, 6) = P(10, 6)/6! = 10!/(4!6!) = 210.$$

Generalizing, if we wish to order N objects, we can view this as three operations: we choose K objects, order them, and then follow with an ordering of remaining $N - K$ objects. The Matching Rule gives

$$N! = C(N, K) K! (N - K)!$$

or

$$C(N, K) = \frac{N!}{K!(N - K)!}.$$

Here are a few interesting observations about $C(N, K)$.

First, $C(N, K) = C(N, N - K)$, which you should verify directly. If we choose K objects from N objects, we are also, in effect, choosing the other $N - K$ objects at the same time. That is, we can view choosing K objects as choosing $N - K$ objects and keeping the K leftovers.

Next, $C(N, K)$ is always a whole number, by definition. If, in the formula $N!/(K!(N - K)!)$, we cancel like terms in $N!$ and $(N - K)!$, we are left with the fraction

$$\frac{N(N - 1)(N - 2) \cdots (N - K + 1)}{K!}.$$

Since this must be a whole number, $K!$ must divide evenly into $N(N - 1)(N - 2) \cdots (N - K + 1)$. A nice way to say this is that $K!$ divides any product of K consecutive whole numbers. For

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Magic Grids

by Ken Fan | edited by Jennifer Sidney

Jasmine: Hey, Emily! What's up?

Emily: I'm thinking about a problem on the latest Math Prize for Girls contest. They didn't hold it this year because of the pandemic, but they still posted problems.

Jasmine: What's the problem?

Emily: It's about something they call a "magic grid." You've got a grid of points, and numbers are assigned to each point. If the numbers assigned to the four corners of any square always add up to the same constant, then the grid is called a magic grid.

Jasmine: Can the four corners be part of a tilted square?

Emily: For the ones they consider, the squares must have horizontal and vertical edges. The problem is about a 6-by-6 magic grid, but I've been wondering about magic grids of other sizes.

Jasmine: I see! It's an amusing variant of the magic square, where the numbers in any row or in any column always add up to the same constant.

Emily: And the numbers along any major diagonal, too... But here, instead of rows, columns, and major diagonals, we're looking at numbers located at the four corners of some square.

Jasmine: What are you thinking about these magic grids?

Emily: I'm wondering how difficult they are to construct.

Jasmine: I see.

Emily: Well, there's not much to think about for 1-by-1 and 2-by-2 grids because you can assign any numbers you wish to the points. For a 1-by-1, there isn't even a constraint; and for 2-by-2, there's just the one constraint that the sum of the four numbers adds up to a constant, which will always be the case.

Jasmine: What about 3-by-3?

Emily: That's just what I was starting to think about when you appeared! Do you want to think about it?

Jasmine: Of course!

Emily: Terrific!

Jasmine: How about we just start assigning variables to the grid points until we run into a constraint?

Emily: Okay. And let's say that the sum of the numbers assigned to the corners of a square is S , for "sum." So S is the "magic sum."

Jasmine: Starting with a blank grid, we can place A in the upper left. To the right of that, we can put B , then put C in the upper right. So far, no constraints.

$$\begin{array}{ccc} A & B & C \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}$$

Emily: We can put D in the left of the middle row. But then for the middle entry, we finally have a constraint, since the four numbers in the upper left 1-by-1 square must sum to S . So we have to put $S - A - B - D$ there.

$$\begin{array}{ccc} A & B & C \\ D & \begin{array}{c} S-A \\ -B-D \end{array} & \cdot \\ \cdot & \cdot & \cdot \end{array}$$

Jasmine: We don't have a choice for the rightmost entry in the middle row, either. In order for the four numbers in the upper right 1-by-1 square to sum to S , we have to put $A + D - C$ there.

$$\begin{array}{ccc} A & B & C \\ D & \begin{array}{c} S-A \\ -B-D \end{array} & \begin{array}{c} A+D \\ -C \end{array} \\ \cdot & \cdot & \cdot \end{array}$$

Emily: Actually, how many constraints are there in total? We've used two so far.

Jasmine: Let's see. There are four 1-by-1 squares, and we've already used the constraints implied by two; then there's the 2-by-2 square. So there are five constraints in total.

Emily: How convenient! There are three unused constraints and there are three more numbers to assign! If the constraints are independent, the values of the missing three numbers are forced.

Jasmine: Let's work it out. If we call the entries in the bottom row X , Y , and Z from left to right, the three constraints correspond to the equations

$$X + Y + D + (S - A - B - D) = S,$$

$$Y + Z + (S - A - B - D) + (A + D - C) = S,$$

and

$$X + Z + A + C = S.$$

Simplifying, we get

$$X + Y = A + B,$$

$$Y + Z = B + C,$$

and

$$X + Z = S - A - C.$$

Emily: As you were writing, I was half expecting the last equation to be $X + Z = A + C$, because the first two equations are consistent with saying that X is A , Y is B , and Z is C !

Jasmine: That's funny.

Emily: In this system, if we add two of the equations and subtract the remaining one, we'll get an equation with just one of the variables X , Y , or Z ! For example, if we add up the first two and subtract the third, we get $2Y = A + 2B + C - (S - A - C) = 2(A + B + C) - S$. So

$$Y = A + B + C - S/2.$$

Jasmine: Nice observation! Using that technique to find X and Z , I get

$$X = S/2 - C$$

and

$$Z = S/2 - A.$$

Emily: So the full grid is

$$\begin{array}{ccc}
 A & B & C \\
 \\
 D & \begin{array}{c} S-A \\ -B-D \end{array} & \begin{array}{c} A+D \\ -C \end{array} \\
 \\
 S/2-C & \begin{array}{c} A+B+ \\ C-S/2 \end{array} & S/2-A
 \end{array}$$

Since we obtained this grid by ensuring that all five equations hold, the variables A , B , C , and D can be assigned any values and the result will be a 3-by-3 magic grid with magic sum S .

Jasmine: To me, it does not look obvious that this is a magic grid. I mean, I know it is, because we just created it. But the fact that any four numbers assigned to vertices of a square sum to S doesn't jump out at me. I have to carry out the computation to verify it. I wonder if it is possible to present the general 3-by-3 magic grid in such a way that it is easier to see that it is, in fact, a magic grid.

Emily: I agree with you. If you handed this grid of formulas to me and told me it was a magic grid with magical sum S , I would have to check all five constraints to convince myself.

Jasmine: Perhaps we should have assigned a single variable to the middle entry instead of to the left entry of the middle row. Maybe if we had done that the formulas would more readily show that the result is a magic grid?

Emily: Let's try it.

Emily and Jasmine assign A , B , and C to the three points in the top row as before, but now assign D to the center of the grid. This time, they get the following formulas for a 3-by-3 magic grid:

$$\begin{array}{ccc}
 A & B & C \\
 \\
 \begin{array}{c} S-A- \\ B-D \end{array} & D & \begin{array}{c} S-B- \\ C-D \end{array} \\
 \\
 S/2-C & \begin{array}{c} A+B+ \\ C-S/2 \end{array} & S/2-A
 \end{array}$$

Jasmine: It's not that different.

Emily: Hey! Maybe we can build up the magic grid from grids that *are* easy to verify as magical. After all, if we have two magic grids, we can add them together and get a new magic grid, though the magic sum would be the sum of the magic sums.

Jasmine: That’s a cool idea. It would be neat if we could find some “basic” magic grids from which all other magic grids can be built. One magic grid that seems pretty basic is the one where all the entries are the same.

Emily: Yes, that would definitely be a magic grid!

Jasmine: Actually, every magic grid G is a sum of two magic grids: one has magic sum 0, and the other is a magic grid with equal entries and magic sum equal to the magic sum of G . This means we can focus on trying to find basic magic grids that have a magic sum of 0.

Emily: I like that plan, and we can use our formula for the general grid to get a formula for the general magic grid with magic sum 0 by setting S to 0. I’ll go ahead and do this in our second magic grid. The formulas simplify to

$$\begin{array}{ccc}
 A & B & C \\
 -A - B - D & D & -B - C - D \\
 =C & A + B + C & -A
 \end{array}$$

Jasmine: What happens if we set each of the variables A , B , C , and D to 1 in turn, with all of the others set to 0?

Emily: We get the four magic grids

$$\begin{array}{ccc|ccc|ccc|ccc}
 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 -1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & -1 & -1 & 1 & -1 \\
 0 & 1 & -1 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0
 \end{array}$$

Jasmine: Those are easier to verify as being magic grids. Having a lot of zeros helps!

Emily: So every magic grid with magic sum 0 can be obtained by multiplying these four magic grids by whichever four numbers we please, then adding them together.

Jasmine: Between the second and fourth, I’m not sure which is easier to verify. The fourth has one more zero, though. And, if we subtract the fourth one from the second, we get a magic grid that looks almost like the 90-degree rotation of the fourth. Since it has more zeros, why don’t we go ahead and replace the second one with the second minus the fourth?

Emily: Okay. Though it’s a really minor point, let’s multiply the fourth by -1 so that it has two 1’s instead of two -1’s.

$$\begin{array}{ccc}
1 & 0 & 0 \\
-1 & 0 & 0 \\
0 & 1 & -1
\end{array}
\qquad
\begin{array}{ccc}
0 & 1 & 0 \\
0 & -1 & 0 \\
0 & 1 & 0
\end{array}
\qquad
\begin{array}{ccc}
0 & 0 & 1 \\
0 & 0 & -1 \\
-1 & 1 & 0
\end{array}
\qquad
\begin{array}{ccc}
0 & 0 & 0 \\
1 & -1 & 1 \\
0 & 0 & 0
\end{array}$$

Jasmine: I wonder if we can replace the first and third with even simpler grids with more zeros.

Emily: The first grid is the only one that would enable us to get a nonzero entry in the upper left corner, and we'd definitely need a grid that has a nonzero entry there. So if we have a 1 in the upper left corner, our formulas tell us that there must be a -1 in the lower right corner. Then, in order for the entries in the corners of the upper left 1-by-1 square to sum to 0, there would have to be at least one nonzero entry in one of the points of the grid marked with an "X":

$$\begin{array}{ccc}
1 & X & \cdot \\
X & X & \cdot \\
\cdot & \cdot & -1
\end{array}$$

Jasmine: If we put the nonzero entry somewhere other than the middle, then we'd have to put another nonzero entry somewhere so that the sum of the numbers assigned to the corners of the bottom right 1-by-1 square also sum to 0; that would result in a grid with at most five zeros, which is what the current basic grid already has. So to avoid that, we'd want to put a nonzero entry in the very middle. Can we make this into a magic grid by adding a single nonzero entry in the middle?

Emily: Unfortunately, no, because for the sum of the numbers assigned to the corners of the upper left 1-by-1 square to be 0, this middle entry would have to be -1; but for the sum of the numbers assigned to the corners of the lower right 1-by-1 square to be 0, the middle entry has to be 1. Impossible!

Jasmine: Then I guess our current four basic grids are about the best we'll be able to do.

Emily: So to sum up what we've found so far, every 3-by-3 magic grid can be written like this:

$$A \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix} + B \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} + C \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} + D \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} S/4 & S/4 & S/4 \\ S/4 & S/4 & S/4 \\ S/4 & S/4 & S/4 \end{pmatrix}$$

Jasmine: On to 4-by-4 magic grids!

To be continued...

Different Angles

by Addie Summer | edited by Jennifer Sidney

Here's a common situation from math class: the teacher presents a mathematical fact and derives it for the class. Then, the class is asked to rederive it for homework or on a test.

Especially during a test, there is time pressure which might cause you to opt to memorize the derivation so that you can quickly reproduce it. But if you forget it during the test, you might find yourself in a very uncomfortable situation. Struggling to recall something – especially under stressful circumstances – is unpleasant, to say the least!

If you find yourself in that situation, try to calm yourself; instead of forcing yourself to recall, let yourself relax and think. Think about the problem. Think of a strategy for solving it.

Maybe, you'll think along lines similar to the derivation you were shown in class. By playing with the ingredients that were used in the derivation, not only do you increase the chance of having your memory jogged, but you might rederive the whole thing yourself.

But even if you do not think along the lines of the derivation presented, you still might succeed in resolving the problem differently – and perhaps in a way that is more natural to you.

Math accommodates many ways of thinking. Have faith that your thoughts are meaningful and can lead to a solution. To illustrate the accommodating nature of mathematics, I'll derive the angle sum formula for the sine function in three different ways.

The Angle Sum Formula for Sine

First, what is the sine function, and what is the angle sum formula for sine? Actually, there are many ways to define the sine function! But since I don't want to dwell on definitions, I'll simply present one way of defining it. Let a be an angle measure. In the coordinate plane, draw a ray emanating from the origin; let the angle the ray creates – as measured counterclockwise from the positive horizontal axis – have measure a . This ray will intersect the unit circle centered at the origin at a point P . The vertical coordinate of P is the sine of a , and is written $\sin(a)$. The horizontal coordinate of P is the cosine of a , and is written $\cos(a)$.

Suppose we know the values of $\sin(a)$, $\cos(a)$, $\sin(b)$, and $\cos(b)$ for two angle measures a and b . These coordinates precisely determine the specific angles a and b (up to multiples of full circles), and these two specific angles have a specific sum $a + b$ (up to multiples of full circles); so the coordinates $(\cos(a + b), \sin(a + b))$ are fully determined by the values $\sin(a)$, $\cos(a)$, $\sin(b)$, and $\cos(b)$. Therefore, it's reasonable to seek a way to compute $\sin(a + b)$ in terms of $\sin(a)$, $\cos(a)$, $\sin(b)$, and $\cos(b)$. In fact, there is a formula for this called the **angle sum formula for sine**. It can be expressed as follows:

$$\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b).$$

This is the formula that we will derive in three different ways. Before reading on, though, please try to derive the formula yourself. And if you figure out a way, press yourself to come up with another way. Be patient. Even if you don't succeed, this will help you understand what follows.

First Derivation

Our first derivation is a very slight variation on an idea I got from a group of 8th graders at the Buckingham, Browne, and Nichols Middle School. See the figure at right.

After drawing in angles that measure a , b , and $a + b$ counterclockwise from the positive horizontal axis, the students observed that angle COB also measures a . This observation enabled them to express the distance BC in two ways to obtain an equation relating relevant quantities. The distance formula applied directly to BC tells us that

$$BC^2 = (\cos(a + b) - \cos(b))^2 + (\sin(a + b) - \sin(b))^2.$$

Because angles COB and AOP have the same angle measure, $BC = AP$. If we apply the distance formula directly to AP , we have

$$AP^2 = (\cos(a) - 1)^2 + (\sin(a) - 0)^2.$$

$$\text{Thus, } (\cos(a + b) - \cos(b))^2 + (\sin(a + b) - \sin(b))^2 = (\cos(a) - 1)^2 + (\sin(a) - 0)^2.$$

If we expand and simplify this last equation, we arrive at

$$\cos(a + b)\cos(b) + \sin(a + b)\sin(b) = \cos(a).$$

(To simplify, we thrice used the identity $\cos^2(x) + \sin^2(x) = 1$, which expresses the fact that $(\cos(x), \sin(x))$ is a point on the unit circle centered at the origin.)

This last formula holds for any values of a and b . Since $\sin(x) = \cos(90^\circ - x)$, let's substitute $90^\circ - (x + y)$ for a , so that the right-hand side becomes $\sin(x + y)$, as well as y for b , so that $a + b = 90^\circ - x$:

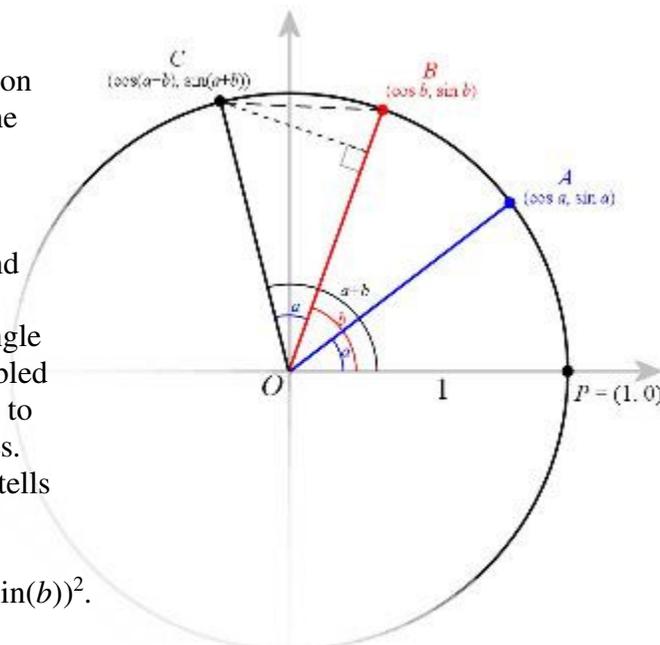
$$\cos(90^\circ - x)\cos(y) + \sin(90^\circ - x)\sin(y) = \cos(90^\circ - (x + y)).$$

Thus,

$$\sin(x)\cos(y) + \cos(x)\sin(y) = \sin(x + y).$$

Beautiful!

Let's review what happened here: the students drew a nice diagram. After indicating the information that they knew, they tried to understand parts of the diagram that they did not know, such as the measure of angle COB . They found that angle to have the same angle measure as angle AOP , and exploited this to compute a length in two different ways.

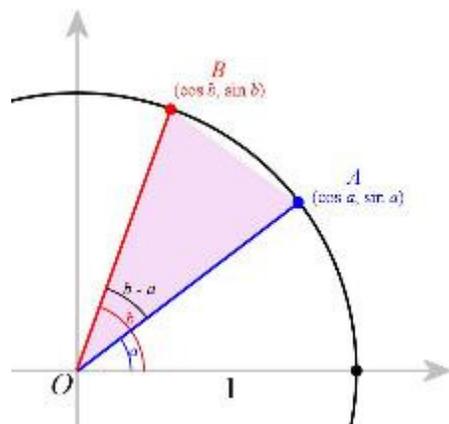


Second Derivation

Suppose you're more fond of areas.

In that case, let's look for an area we can compute that involves the relevant quantities. For example, the shaded triangle, triangle ABO , has coordinates involving the sines and cosines of a and b , and is an isosceles triangle with apex angle $b - a$.

Let's compute the area of this triangle in two ways.



For the first way, we'll use the formula $\frac{1}{2}xy \sin \theta$ for the area of a triangle with two sides of length x and y and included angle θ . Here, we'll take the two sides of length 1 (which are both radial lines of the circle) and their included angle, which measures $b - a$, to obtain the formula

$$\frac{1}{2} \sin(b - a).$$

For the second way, we'll subtract the area of three right triangles from the indicated rectangle in the figure at right to leave us with the area of triangle ABO .

The rectangle has width $\cos(a)$ and height $\sin(b)$, so its area is $\cos(a)\sin(b)$.

For the three right triangles labeled I, II, and III in the figure, we can use the formula $1/2$ base times height, because the legs of a right triangle can be conveniently taken to be a base and height.

Triangle I has area $\cos(b)\sin(b)/2$.

Triangle II has area $(\cos(a) - \cos(b))(\sin(b) - \sin(a))/2$.

Triangle III has area $\cos(a)\sin(a)/2$.

Therefore, the shaded triangle ABO has area

$$\cos(a)\sin(b) - (\cos(b)\sin(b) + (\cos(a) - \cos(b))(\sin(b) - \sin(a)) + \cos(a)\sin(a))/2,$$

which simplifies to $(\sin(b)\cos(a) - \cos(b)\sin(a))/2$. Equating these two formulas for area and multiplying throughout by 2, we find that

$$\sin(b - a) = \sin(b)\cos(a) - \cos(b)\sin(a).$$

If we replace a with $-a$ throughout (and use the identities $\sin(-a) = -\sin(a)$, $\cos(-a) = \cos(a)$) and rearrange terms, we obtain the angle sum formula for sine:

$$\sin(a + b) = \sin(a)\cos(b) + \sin(b)\cos(a).$$

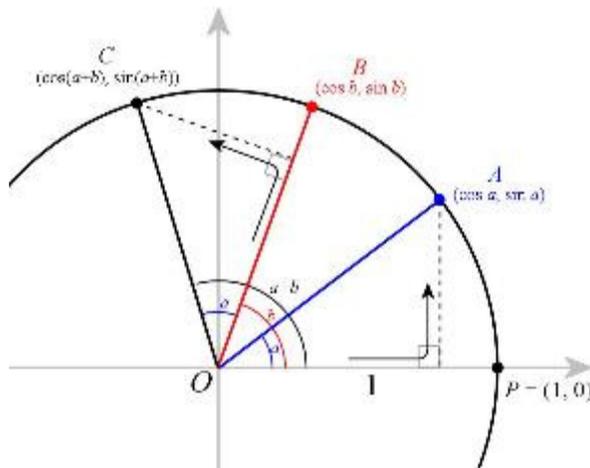
Third Derivation

Maybe you enjoy walking. So, let's take a walk.

We begin our walk at the origin of coordinates facing right, straight down the positive horizontal axis.

We walk $\cos(a)$ units, turn left 90° , then walk another $\sin(a)$ units. Where do we end up?

We end up at point A , since its coordinates are $(\cos(a), \sin(a))$.



Let's do this walk again, starting at the origin but this time looking in the direction of B . We walk forward $\cos(a)$ units, turn left 90° , then walk another $\sin(a)$ units. Now we end up at the point C . Why? Because we can rotate the figure around O so that the ray pointing from O to B becomes the positive horizontal axis. (Or, tilt your head to the left.) Suddenly, C becomes the point on the unit circle rotated through an angle of measure a from the positive horizontal axis!

Since B is on the unit circle, if we walk $\cos(a)$ units from O to B , we end up at the point whose coordinates can be found by multiplying the coordinates of B by $\cos(a)$:

$$(\cos(a)\cos(b), \cos(a)\sin(b)).$$

When we turn left by 90° , we are facing in the direction from O to $(-\sin(b), \cos(b))$ (because the point $(-\sin(b), \cos(b))$ is the counterclockwise 90° rotation of $B = (\cos(b), \sin(b))$ around O). Therefore, if we walk in this new direction a total of $\sin(a)$ units, our displacement can be found by multiplying the coordinates $(-\sin(b), \cos(b))$ by $\sin(a)$.

Thus, we end up at the point

$$(\cos(a)\cos(b) - \sin(a)\sin(b), \cos(a)\sin(b) + \sin(a)\cos(b)).$$

Since C also has coordinates $(\cos(a + b), \sin(a + b))$, we get the angle sum formulas for both sine and cosine by equating corresponding components:

$$\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\sin(a + b) = \cos(a)\sin(b) + \sin(a)\cos(b).$$

Summary

There are many other derivations of the angle sum formula for sine. So don't feel that you have to memorize a particular derivation. If you play around with this material, you will surely find others; if you'd like to share it, please send yours to girlsangle@gmail.com!

Member's Thoughts

On a Theorem of Milena Harned

by Ken Fan | edited by Amanda Galtman

For eons, kites with three congruent acute angles longed to be recognized for their special property. Their hopes ran high during the golden age of Euclidean geometry. They watched as a parade of shapes marched by, each showered with adoration for their marvelous properties: right triangles, isosceles triangles, acute triangles, obtuse triangles, scalene triangles, squares, rectangles, rhombi, trapezoids, circles, parabolas, ellipses, hyperbolas, tetrahedra, cubes, octahedra, dodecahedra, icosahedra, cylinders, spheres ... a geometric menagerie!

Alas, they were neglected.

Two thousand years would pass before, at long last, the 13-year-old Milena Harned discovered their secret: that they, alongside rhombi, are the only convex quadrilaterals whose angle bisectors also bisect their perimeters.

Let's think about this in more detail to gain an appreciation for Milena's theorem.

At each vertex of a convex quadrilateral, there is an angle bisector, which is a line that splits the angle in that corner exactly in half. This angle bisector splits the polygon itself into two parts. For which convex quadrilaterals do these angle bisectors chop the boundary of the polygon into two pieces of the same length?

Triangles

Please take out some scratch paper and try to show that the only triangles whose angle bisectors are perimeter bisectors are equilateral. We can see by symmetry that an equilateral triangle enjoys the property. It takes more effort to show that equilateral triangles are the only ones. Non-equilateral isosceles triangles have one angle bisector that bisects its perimeter, but the equilateral triangles are the only ones where all three angle bisectors are perimeter bisectors.

Convex Quadrilaterals

Suppose $ABCD$ is a convex quadrilateral. For now, let's focus on understanding when just one angle bisector is a perimeter bisector. We'll focus on the angle bisector at the vertex A .

Let X be the point (other than A) where the angle bisector at A intersects the boundary of $ABCD$.

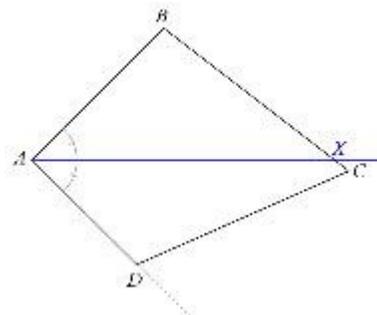
First of all, where can X be located? Because X is on the angle bisector at A , it cannot be located on the sides AB or AD which form the angle at A , so X must be in the interior of BC or CD , or at the vertex C itself. Without loss of generality, we may assume that X is in the interior of side BC or at C because we can relabel our diagram, swapping the labels " B " and " D ", if necessary.

If we are lucky enough that X coincides with C , then we know that $AB + BC = AD + DC$. In other words, B and D are on some ellipse with foci at A and C . And because the angles BAC and DAC are congruent, by the mirror symmetry of an ellipse, we may conclude that $ABCD$ is a kite.

We still need to figure out which kites, if any, have angle bisectors at B and D that also bisect their perimeters, and, after determining which kites, we need to go even further and show that those kites are the *only* such convex quadrilaterals.

In other words, if X does *not* coincide with C , does it still follow that $ABCD$ is a kite if all of its angle bisectors are perimeter bisectors? The perimeter bisection condition tells us that $AB + BX = AD + DC + CX$. Is this situation even possible?

Here's a figure to illustrate our situation. We want to know if there are quadrilaterals $ABCD$ where the angle bisector at A meets the interior of side BC at X and $AB + BX = AD + DC + CX$. It's not difficult to arrange that the angle bisector at A intersects the interior of BC at a point X . So let's start with such a quadrilateral, and if $AB + BX$ is not equal to $AD + DC + CX$, let's see if we can modify it so that $AB + BX = AD + DC + CX$.



Let's think of D as moveable and slide it along the ray that starts at A and maintains a constant angle XAD equal to that of angle XAB . When D constrains its movement to that ray, the segment AX remains an angle bisector at A that intersects the perimeter of our quadrilateral at X . As we move D , the quantity $AB + BX$ does not change, but the quantity $AD + DC + CX$ does. As D gets farther from A , the quantity $AD + DC + CX$ tends to infinity. When D coincides with A , this quantity becomes $AC + CX$. Thus, if $AC + CX < AB + BX$, there is at least one location for D where our angle bisector AX is a perimeter bisector, too. The condition $AC + CX < AB + BX$ is equivalent to C being situated inside the ellipse that passes through B and has foci at A and X .

So, it is definitely possible to create a quadrilateral $ABCD$ where the angle bisector at A is also a perimeter bisector and intersects the boundary at a point X that is interior to side BC . According to our analysis, we can construct one as follows: First, we fix some angle measure $\theta < 90^\circ$ and draw a horizontal line segment AX , which will represent our angle and perimeter bisector. We then draw some point B so that angle BAX measures θ . We then draw the ellipse through B with foci at A and X . Next, we draw the line segment BX and extend this beyond X to some arbitrary point C still inside the ellipse. We then find a point D for which angle XAD measures θ and $AB + BX = AD + DC + CX$. If the resulting quadrilateral $ABCD$ is convex, we have an example. The property that angle bisectors are perimeter bisectors does not depend on scale, so we might as well assume that $AX = 1$. Even so, the choice of θ represents 1 degree of freedom, the location of B has 1 degree of freedom, and the choice of C has 1 degree of freedom. This construction suggests that the set of convex quadrilaterals having an angle bisector that is also a perimeter bisector, is a 3D space. (Quadrilaterals offer much more diversity than triangles!) For each example, we could then check if its other angle bisectors are perimeter bisectors. We know that the perimeter bisector that has one endpoint at C must intersect AB at a point Y such that $AY = CX$ (why?), so one of these conditions would be that the angles BCY and DCY are congruent, but how would we translate this condition into clear restrictions on θ , B , and C ? If you introduce coordinates for B and C and write down the equation that says these two angles are congruent, you will be looking at a rather complicated equation in at least 3 variables! And there still remain to be checked the angle bisectors at vertices B and D ! What can be done?

Well, if you're stuck, it's all ingeniously solved in Milena's paper: "Perimeter Bisectors, Cusps, and Kites" in the peer-reviewed International Journal of Geometry, Volume 10, Number 4: <https://ijgeometry.com/wp-content/uploads/2021/09/6.-85-106.pdf>. Check it out!

Calendar

Session 29: (all dates in 2021)

September	9	Start of the twenty-ninth session!
	16	
	23	
	30	
October	7	
	14	
	21	
	28	
November	4	
	11	Karia Dibert, University of Chicago
	18	
	25	Thanksgiving - No meet
December	2	

Session 30: (all dates in 2022)

January	27	Start of the thirtieth session!
February	3	
	10	
	17	
	24	No meet
March	3	
	10	
	17	
	24	No meet
	31	
April	7	
	14	
	21	No meet
	28	
May	5	

Girls' Angle has been hosting Math Collaborations at schools and libraries. Math Collaborations are fun math events that can be adapted to a variety of group sizes and skill levels. We will soon have versions available that are designed for remote participation. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

Girls' Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) _____ (first) _____

Parents/Guardians: _____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: _____ Cell Phone: _____

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

The \$50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- Enclosed is a check for \$50 for a 1-year Girls' Angle Membership.
- I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.



A Math Club for Girls

Girls' Angle Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is \$20/meet for members and \$30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle is based in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, founder and director of the Exploratory
Yaim Cooper, Institute for Advanced Study
Julia Elisenda Grigsby, professor of mathematics, Boston College
Kay Kirkpatrick, associate professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, assistant dean and director teaching & learning, Stanford University
Lauren McGough, postdoctoral fellow, University of Chicago
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, associate professor, University of Utah School of Medicine
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Liz Simon, graduate student, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, associate professor, University of Washington
Karen Willcox, Director, Oden Institute for Computational Engineering and Sciences, UT Austin
Lauren Williams, professor of mathematics, Harvard University

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) _____ (first) _____

Parents/Guardians: _____

Address: _____ Zip Code: _____

Home Phone: _____ Cell Phone: _____ Email: _____

Please fill out the information in this box.

Emergency contact name and number: _____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

(Parent/Guardian Signature) Date: _____

Participant Signature: _____

Members: Please choose one.

- Enclosed is \$216 for one session (12 meets)
- I will pay on a per meet basis at \$20/meet.

Nonmembers: Please choose one.

- I will pay on a per meet basis at \$30/meet.
- I'm including \$50 to become a member, and I have selected an item from the left.

I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

**Girls' Angle: A Math Club for Girls
Liability Waiver**

I, the undersigned parent or guardian of the following minor(s)

_____ ,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: _____ Date: _____

Print name of applicant/parent: _____

Print name(s) of child(ren) in program: _____