

Girls' *Angle* Bulletin

December 2020/January 2021 • Volume 14 • Number 2

To Foster and Nurture Girls' Interest in Mathematics



An Interview with Pamela E. Harris, Part 2
The Needell in the Haystack: Questions,
Lyme Disease Data, and More Questions, Part 2
Toblerone Game

Honk! Honk!, Part 1
The Saga of Fran & Fred
You Do Math
Notes from the Club

From the Founder

At Girls' Angle, we believe in learning by doing, and preferably, doing math of member's own creation. Achieving this has definitely been harder during the pandemic. We're super proud of all our members and mentors for keeping the math alive. - Ken Fan, President and Founder

Girls' Angle Donors

A Heartfelt Thank You to our Donors!

Individuals

Uma Achutha	Mark and Lisel Macenka
Dana Albert	Brian and Darline Matthews
Nancy Blachman and David desJardins, founders of the Julia Robinson Mathematics Festival, jrmf.org .	Toshia McCabe
Bill Bogstad	Mary O'Keefe
Ravi Boppana	Stephen Knight and Elizabeth Quattrochi Knight
Lauren Cipicchio	Junyi Li
Merit Cudkowicz	Alison and Catherine Miller
Patricia Davidson	Beth O'Sullivan
Ingrid Daubechies	Robert Penny and Elizabeth Tyler
Anda Degeratu	Malcolm Quinn
Kim Deltano	Jeffrey and Eve Rittenberg
Concetta Duval	Craig and Sally Savelle
Glenn and Sara Ellison	Eugene Shih
John Engstrom	Eugene Sorets
Lena Gan	Sasha Targ
Jacqueline Garrahan	Diana Taylor
Courtney Gibbons	Waldman and Romanelli Family
Shayne Gilbert	Marion Walter
Vanessa Gould	Andrew Watson and Ritu Thamman
Rishi Gupta	Brandy Wieggers
Larry Guth	Brian Wilson and Annette Sassi
Andrea Hawksley	Lissa Winstanley
Scott Hilton	The Zimmerman family
Delia Cheung Hom and Eugene Shih	Anonymous
David Kelly	

Nonprofit Organizations

Draper Laboratories
The Mathenaem Foundation
Orlando Math Circle

Corporate Donors

Adobe
Akamai Technologies
Big George Ventures
D. E. Shaw
John Hancock
Maplesoft
Massachusetts Innovation & Technology Exchange (MITX)
Mathenaem
MathWorks, Inc.
Microsoft
Microsoft Research
Nature America, Inc.
Oracle

For Bulletin Sponsors, please visit girlsangle.org.

Girls' Angle Bulletin

*The official magazine of
Girls' Angle: A Math Club for girls
Electronic Version (ISSN 2151-5743)*

Website: www.girlsangle.org
Email: girlsangle@gmail.com

This magazine is published six times a year by Girls' Angle to communicate with its members and to share ideas and information about mathematics.

Girls' Angle welcomes submissions that pertain to mathematics.

The print version of the Bulletin is printed by the American Mathematical Society.

Editors: Amanda Galtman
Jennifer Sidney Silva
Executive Editor: C. Kenneth Fan

Girls' Angle: A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

FOUNDER AND PRESIDENT

C. Kenneth Fan

BOARD OF ADVISORS

Connie Chow
Yaim Cooper
Julia Elisenda Grigsby
Kay Kirkpatrick
Grace Lyo
Lauren McGough
Mia Minnes
Bjorn Poonen
Beth O'Sullivan
Elissa Ozanne
Katherine Paur
Liz Simon
Gigliola Staffilani
Bianca Viray
Karen Willcox
Lauren Williams

On the cover: *Honk! Honk!* by C. Kenneth Fan. See *Honk! Honk!, Part 1* on page 15.

An Interview with Pamela E. Harris, Part 2

This is the concluding half of our interview with Associate Professor of Mathematics at Williams College, Pamela E. Harris.

The secret to success in mathematics is all a simple equation of interest, hard work, and curiosity.

Ken: Please tell us about some of the people who were influential to you in your journey to becoming a mathematician.

Pamela: I've mentioned Alejandra and Erik. Another influential person has been my PhD advisor Jeb Willenbring who has always been very supportive of my career choices and he encouraged me to pursue them fully. Even when I failed my masters exams (multiple times) and others in the faculty weren't so supportive, Jeb always reminded me that I could do it. He spent so many hours with me working on math and sharing his love for the subject. I have fond memories of hanging out in his kitchen as Charlotte (his daughter who was a toddler at the time) banged on pots and pans all while Jeb would tell me the history of math theorems and of interesting problems yet unsolved. Jeb has also overcome a lot and his own life story gives me so much inspiration. I'm tearing up thinking of it because I may never tell him directly (yikes he may read this!), but I am proud to forever be called his student.

Ken: I'll make sure he has a copy! You raised a child as a graduate student, which is utterly amazing to me. How did you do it?

Pamela: Akira was born in April and I started graduate school that August, both occurred while Jamual (my husband) was deployed to Iraq. In fact Jamual was stationed out of state during my entire graduate schooling (6 years in total). Since I started graduate school as a mom I never knew what the experience of being a childless grad student was like. So I just assumed we all had it hard and we all would work to help each other stay up to date with our material. I was very fortunate that UWM supported me with a wonderful fellowship that helped me cover expenses associated with caring for Akira and my mother helped me during the first three years of my grad program. Then as soon as I was able to, I enrolled Akira in a full day K3 school. That was emotionally painful to do because she was so little. I remember her carrying a book bag that barely cleared the ground as she walked in to the school. I spent many days crying in the car after she went in, often second guessing the choices I was making. But I pushed through and we got through the next three years. I studied when she slept and woke up earlier than her to make sure I could take care of things she needed for the day. I took it one day at a time and reminded myself that this would be worth it in the end.

Ken: Do you have any advice for how to best approach learning mathematics?

Pamela: Be curious and ask questions constantly. Always read things and do not be discouraged if new words appear. Simply make a list and learn new words daily. Math is like any language. Once you know the words you can then start with sentence construction. Once you understand

Dear Reader,

We're committed to producing quality math educational content and make every effort to provide this content to you for free.

We are also committed to surviving as a nonprofit!

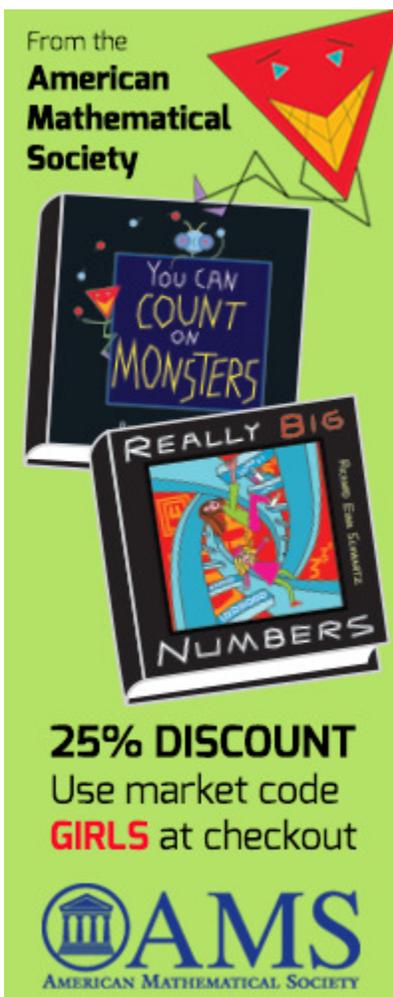
For this issue, those who do not subscribe to the print version will be missing out on a portion of this interview with Prof. Pamela E. Harris and other content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

We know that mathematical interest and talent is unrelated to economic status, which is why we provide so much content for free. But we hope that those of you who are comfortable financially will help us to continue in our efforts.

So, please consider subscribing to the Bulletin. Thanks to our sponsors, subscriptions cost \$36/year. With a subscription, you have also gained access to our mentors via email and the ability to influence content in this Bulletin. Visit www.girlsangle.org/page/bulletin_sponsor.html for more information.

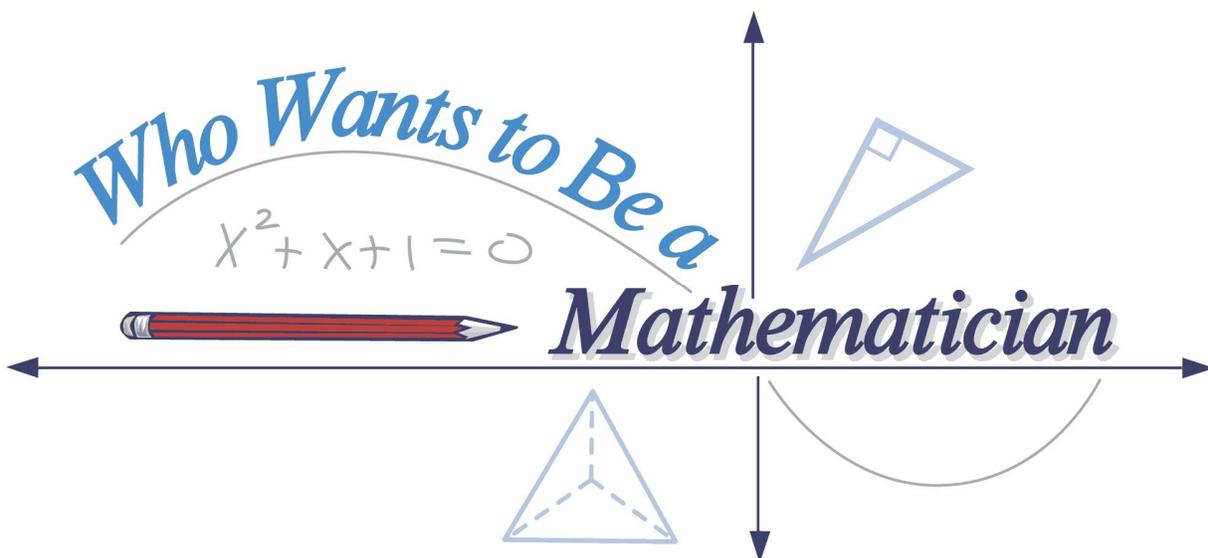
Thank you and best wishes,
Ken Fan
President and Founder
Girls' Angle: A Math Club for Girls

Content Removed from Electronic Version



The American Mathematical Society is generously offering a 25% discount on the two book set *Really Big Numbers* and *You Can Count On Monsters* to readers of this Bulletin. To redeem, go to <http://www.ams.org/bookstore-getitem/item=MBK-84-90> and use the code “GIRLS” at checkout.

Content Removed from Electronic Version



America's Greatest Math Game: Who Wants to Be a Mathematician.

***k*-Nearest Neighbors** In Figure 5 we display the results of running the KNN model on individual features. To do this, we train our model using only a single feature in place of the entire data set. We can interpret this as projecting each data point onto the axis of that feature, so the space we are learning within is one-dimensional. For each feature, we tune the model on a small batch of possible *k* values.

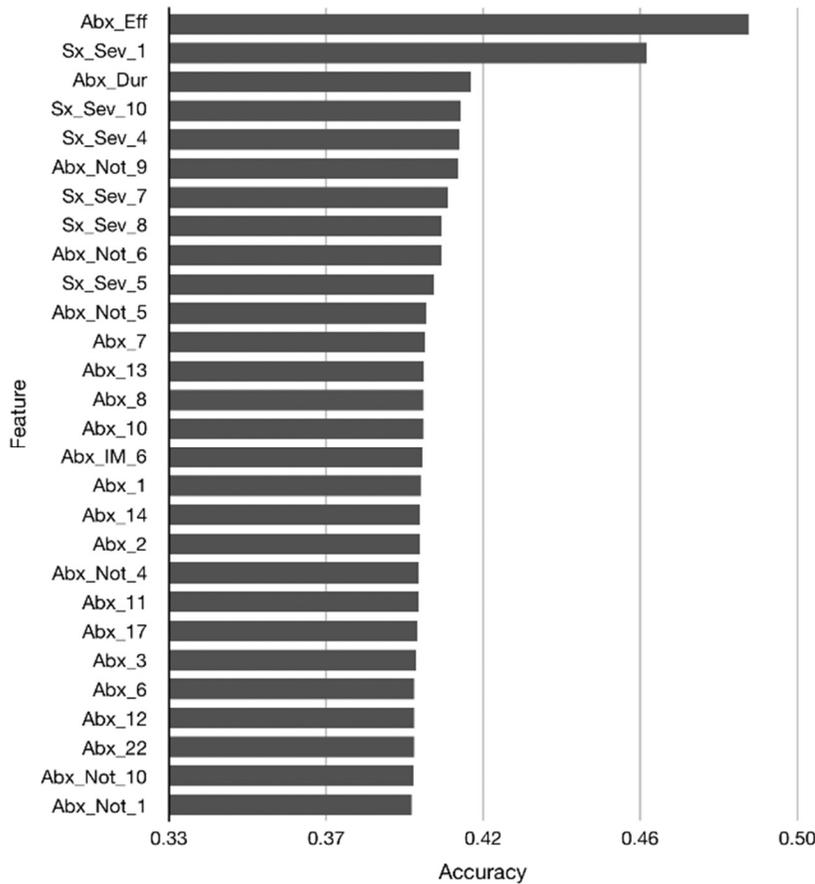


Figure 5. KNN accuracy of individual features on Subsampled MLD (SMLD). Here we display the top 30 features by accuracy, in descending order.

Putting it all together From the results of our models we now create a ranking of the most important features (questions in the survey) in our dataset for predicting GROC labels. To do this, we first define $R(m, i)$ to be the ranking of feature i by model m . Since we have 215 features, note that $1 \leq R(m, i) \leq 215$ for all i, m . For each, the metric we use to produce the ranking is the metric used to order the features in Figures 1-5. In order to aggregate these rankings, we take the simple approach of averaging the ranking of each feature by all of our models. Let $S(i)$ be the average rank of feature i . Then

$$S(i) = \frac{1}{5} \sum_{m=1}^5 R(m, i).$$

This aggregates our rankings into a single score where smaller values indicate more important features. In Table 2, we show the top 30 features sorted by value. Note that there are many quantitative ways to measure the importance of these features, but we omit describing them in this article. That topic could easily take up its very own article.

Content Removed from Electronic Version

Girls!

Learn Mathematics!



Make new Friends!

Meet Professional Women who use math in their work!



Improve how you Think and Dream!

Girls' Angle

A math club for ALL
girls, grades 5-12.

girlsangle@gmail.com
girlsangle.org

Girls' *Angle*

Toblerone Game

by Eleanor Bogosian, Antonella Catanzaro,
Elizabeth Cavatorta, and Naomi Danison | edited by Amanda Galtman

Primary guidance and mentorship for this mathematical investigation was provided by Laura Pierson.



Photo by Evan-Amos

A Toblerone bar, made by Kraft Foods.

Introduction

Two friends share a Toblerone candy bar. A Toblerone candy bar is a chocolate bar formed by connecting triangular pieces of chocolate. Each wants as many of the triangles as possible, but they don't want to rudely gobble down the whole bar in one go. They decide to split the bar by playing a game, whose rules are described below. In this article, we determine how to play this game optimally.

Definitions

Triangle	A triangular piece of chocolate in a Toblerone candy bar.
Segment	A Toblerone bar consisting of any number of triangles.
Even segment	A segment with an even number of triangles.
Odd segment	A segment with an odd number of triangles.
Singleton	A segment with exactly one triangle
P1	Player one.
P2	Player two.
Score	The number of triangles taken by P1 minus the number of triangles taken by P2.

Rules of the Toblerone Game

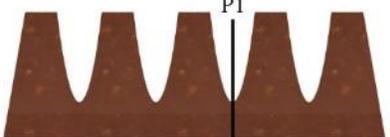
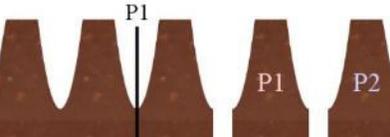
There are two players, P1 and P2. They take turns either taking a singleton (if one is available) or splitting a segment (if one is available with more than two triangles). P1 goes first.

At the start of a game, there can be any number of bars with any number of triangles in each. If there is at least one singleton, the player can either take one or split a segment (if one is available). If there are no singletons but there are segments, then a segment must be split into two new segments. The player can choose to split a segment between any two adjacent triangles.

The game ends when there are no more triangles left. The winner is the player with the most triangles at the end of the game.

Sample Game

Here's an example of a game where the second player wins.

Action	Game State	P1	P2
Starting configuration		0	0
Player 1 splits between the 3 rd and 4 th triangles		0	0
Player 2 splits between the 4 th and 5 th triangles		0	0
Player 1 takes the 4 th triangle		1	0
Player 2 takes the 5 th triangle		1	1
Player 1 splits between the 2 nd and 3 rd triangles		1	1
Player 2 takes the 3 rd triangle		1	2
Player 1 splits between the 1 st and 2 nd triangles		1	2
Player 2 takes the 2 nd triangle		1	3
Player 1 takes the 1 st triangle		2	3

Optimal Play

We first prove the following lemma.

Lemma. *No matter which segment is split, there is always a change in the parity of the number of even segments and no change in the parity of the number of odd segments.*

Proof. If an even segment is split, there are two possibilities. The first possibility is that two even segments are created. In that case, there is one more even segment, so the parity of the number of even segments changes. The second possibility is that two odd segments are created. In that case, there is one less even, so there is again a change in the parity of the number of even segments. Both possibilities result in no change in the parity of the number of odd segments. If an odd segment is split, there is only one possibility: an odd segment is always split into an even segment and an odd segment. There is one more even segment, so there is yet again a change in the parity of the even segments. There is no change in the number of odd segments. □

Theorem. *Given the parity of the numbers of singletons, even segments, and odd segments, one can determine the final score of an optimally played game according to the table below.*

Table of Optimal Scores

State Label	Parities			Final Score	Next State
	Singletons	Evens	Odds		
A	0	0	0	0	F C G
B	0	0	1	-1	E D H
C	0	1	0	0	H A (E)
D	0	1	1	1	G B (F)
E	1	0	0	2	B (G) (C)
F	1	0	1	1	A (H) (D)
G	1	1	0	0	D (E) A
H	1	1	1	1	C (F) B

This table shows the final score for each possibility given the parities of the numbers of singletons, evens, and odds in the starting state. A '1' represents an odd amount and a '0' represents an even amount. The 'Next State' column shows the only states the game can move to next, listed in the following order: taking a singleton, splitting a segment without creating a singleton, and splitting a segment while creating a singleton. Suboptimal choices are enclosed in parentheses.

Proof. The proof will be a proof by induction with one base case and eight different cases in the inductive step. We will be inducting on the number of moves in the game. (Note that the total number of moves in a game is equal to the total number of triangles and adjacent pairs of triangles in all the segments in the starting configuration.)

Base Case

The base case is when there are no triangles, so the score is 0.

Inductive Step

In some cases, a player has a choice between states. Because the proof concerns optimal gameplay, we always assume that the player chooses the optimal state.

To verify the final score the current state produces, we look at the final score the next state produces, switch the sign, and add 1 if the player takes a singleton in this turn. When the game moves to the next state, the score switches signs, because P1 and P2 switch places.

In state A, if a player takes a singleton, state A becomes state F, with a score of 1. If the player splits a segment instead, depending on whether a singleton is created, state A becomes state C or state G. Both have a score of 0. In all cases, state A leads to a final score of 0.

In state B, if a player takes a singleton, state B becomes state E, which produces a final score of 2. If a player splits a segment, state B becomes state D or state H, which both have a score of 1. In all cases, state B leads to a final score of -1.

In state C, if the player takes a singleton, state C becomes state H, with a score of 1. If a player splits a segment, state C becomes state A or state E, which has a score of 0 or 2, respectively. The number of even segments is odd, so there must be an even segment. When splitting an even, one can choose whether to create an odd number of singletons. State A gives state C a score of 0, while state E gives state C a score of -2. In optimal gameplay, state C becomes state A or state H. State A and state H both give state C a score of 0.

In the same manner as state C, state D becomes state G, state B or state F. These next states have scores of 0, -1 and 1, respectively. In optimal gameplay, state C becomes state G or state B, which both give state D a score of 1.

In state E, if the player takes a singleton, state E becomes state B, with a score of -1. If the player splits a segment, state E becomes state G or state C, which both have a score of 0. State B gives state E a score 2, while state G and state C both give state E a score of 0. State B is optimal, so the score for state E is 2.

In state F, if the player takes a singleton, state F becomes state A, with a score of 0. If the player splits a segment, state F becomes state H or state D. Both have a score of 1. State A gives state F a score of 1, while state H and state D give state F a score of -1. State A is optimal, so the score for state F is 1.

In state G, if the player takes a singleton, state G becomes state D, with a score of 1. If the player splits a segment, state G becomes state E or state A. These next states have scores of 2 and 0, respectively. State D and state A give state G a score of 0, while state E gives state G a score of -2. State D and state A are equally optimal, so the score for state G is 0.

In state H, if the player takes a singleton, state H becomes state C, with a score of 0. If the player does not, then it will become state F or state B, which have a score of 1 and -1, respectively. State C and state B give state H a score of 1, while state F gives state G a score of -1. State C and B are equally optimal, so the score for state H is 1.

This exhausts all the possibilities and completes the inductive step and the proof. \square

Conclusion

There are usually many possibilities for each step of the game, but they can all be reduced to eight different states based on the parity of the singletons, evens, and odds. The state at the start of the game determines the final score. Here are a few interesting observations we made:

- Taking a singleton is always an optimal move, although not necessarily the only one.
- Unless a singleton is created, which bar is split and the way it is split do not impact the outcome of the game.
- An optimal score of 2 is possible, not just -1, 0, or 1. See state E

Honk! Honk!, Part 1¹

An Introduction to Parking Functions

by Kimberly P. Hadaway and Pamela E. Harris²

You have seen a car trying to park before, and that is the perfect introduction to **parking functions**! Imagine that we are looking at a one-way street with a dead end, and there is parking available on only one side of the street. Suppose there are n parking spots available on the street, and they are labeled 1 to n from the start of the street to the end in consecutive order. We illustrate this street in the figure below.

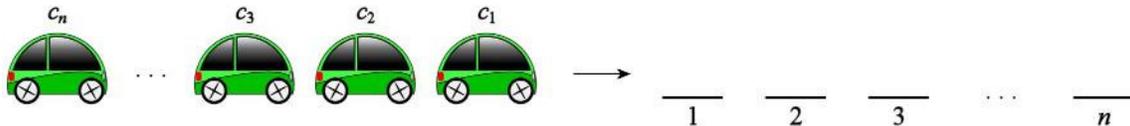


Figure 1. Parking function illustration

As part of this scenario, we have a line of n cars waiting to park. Like most drivers, each of these drivers has a favorite parking spot. We call each driver's favorite spot their **preference**. Note that preferences are not required to be distinct; that is, multiple drivers can have the same favorite parking spot. In order to keep track of the parking preferences of each driver, we list them in a vector in the same order that the cars are lined up to park; that is, the i th car will have its preference listed in the i th spot of this vector, which we call a **preference vector**. The preference vector is important because it allows us to easily document each car's favorite spot.

In this scenario, we want each car to be able to park, using the following **parking rule**: Cars enter the street in the order they are lined up. Since we know each car has a desired parking spot, each car drives down the street to this spot and checks if it is available. If the spot is empty, the car parks in that spot – hooray! Then, the next car enters the street and repeats the process. If instead the car found its favorite spot occupied, then the car continues down the street and parks in the next available spot. We proceed in this way until either a car reaches the dead end and cannot park (sad!), or all cars are able to park. If a preference vector allows all cars to park using this parking rule, then we call it a **parking function**.

Keep in mind that not all preference vectors will allow all of the cars to park. This happens because the street has a dead end. For example, if a car's preferred spot and all of the following parking spots are taken, then the car can't park – based on our parking rule – so this particular preference vector would not be a parking function. Let's consider two quick examples. The parking preference vector $(1, 1, 4, 2)$ is a parking function because car 1 parks in spot 1; car 2 finds spot 1 already occupied and parks in spot 2; car 3 parks in its preferred parking spot numbered 4; car 4 finds its preferred parking spot occupied and parks in spot 3. The parking preference vector $(2, 2, 3, 4)$ is not a parking function. Notice car 1 parks in spot 2; car 2 finds its preferred parking spot occupied and parks in spot 3; car 3 finds its preferred parking spot occupied and parks in spot 4; car 4 reaches its preferred parking spot which is occupied and now encounters the dead end. Since not all cars could park, we have established that $(2, 2, 3, 4)$ is not a parking function.

There are many preference vectors that do allow all of the cars to park. For example, any permutation provides a parking function. Recall that a permutation is an arrangement of objects

¹ This content supported in part by a grant from MathWorks.

² Both authors are from the Department of Mathematics and Statistics at Williams College.

where the order matters. For example, if the objects we can arrange are x , y and z , then we can arrange them in 6 different ways:

$$(x, y, z), (x, z, y), (y, x, z), (y, z, x), (z, x, y), \text{ and } (z, y, x).$$

Lemma 1. There are $n!$ permutations of the numbers $1, 2, \dots, n$, and they are all parking functions.



Try to prove Lemma 1 before reading further.

Proof. First, we show that there are $n!$ permutations. The first position could contain any of the n numbers $1, 2, \dots, n$. Once we place a number in the first position, then there are $n - 1$ choices of numbers we can place in the second position, and so on until we have only one remaining number to place in the last position. This means that the total number of ways we could arrange the numbers $1, 2, \dots, n$ is given by $n(n - 1)(n - 2) \cdots 3 \cdot 2 \cdot 1 = n!$.

Next, we show that every permutation is a parking function. This requires that we think of a permutation as a parking preference vector. Notice since every car has a unique favorite parking spot, then when they enter the street, not only can they park, but they all also get their preferred spot. This means that every permutation is in fact a parking function, as we claimed. \square

Lemma 1 establishes that all of the permutations are parking functions, but what about a vector like $(1, 1, 1, 1)$? This is a parking function, but it is not a permutation. So, we know that there must be more than $n!$ parking functions, but we do not yet know how many parking functions there are. Hence, we ask: *If n cars enter the one-way street with n parking spots, how many parking functions are there?* Let's continue exploring to discover the answer!

1. When is a preference vector a parking function?

As mathematicians, we are often interested in finding answers to new things. So, we could start by counting how many preference vectors there are.



Can you figure it out before reading on? How many preference vectors are there?

Lemma 2. There are n^n preference vectors where n represents the number of cars and the number of parking spots.

Proof. Recall that there are n entries in each vector, representing the favorite parking spots of the n cars entering the street to park. Since we have n parking spots, each car has n choices for its preference. So, we have n choices for the first entry in the vector, n choices for the second entry in the vector, n choices for the third entry, and so on until the n th position in the vector. Thus, n^n is the number of distinct preference vectors, as we claimed. \square

Next, we are interested in counting how many of the n^n preference vectors are parking functions. We know from Lemma 1 that there are at least $n!$ parking functions. In fact, we know that there must be more than $n!$ because of the parking function $(1, 1, 1, \dots, 1)$, which is not a permutation.



Can you figure out exactly how many parking functions there are? If you can't, try to work out some examples and guess the formula.

Let's work out a small example.

Example 1. Let's consider $n = 3$.

We already know there are $3^3 = 27$ preference vectors.

 Grab a piece of scratch paper and list all 27 preference vectors. When you're done, check your answer against our list below.

(1, 1, 1)	(1, 1, 2)	(1, 1, 3)	(1, 2, 1)	(1, 2, 2)	(1, 2, 3)	(1, 3, 1)	(1, 3, 2)	(1, 3, 3)
(2, 1, 1)	(2, 1, 2)	(2, 1, 3)	(2, 2, 1)	(2, 2, 2)	(2, 2, 3)	(2, 3, 1)	(2, 3, 2)	(2, 3, 3)
(3, 1, 1)	(3, 1, 2)	(3, 1, 3)	(3, 2, 1)	(3, 2, 2)	(3, 2, 3)	(3, 3, 1)	(3, 3, 2)	(3, 3, 3)

Now, we can check each of these preference vectors to see which ones allow all three cars to park. Together, let's determine whether (2, 2, 1) and (3, 2, 3) are parking functions. In the case of (2, 2, 1) we have the following occurring:

- The first car enters the street and parks in its preferred spot numbered 2.
- The second car enters the street and goes to the second parking spot (its preferred spot) but finds that car 1 already parked there! So, it continues down the street and parks in the next available parking spot, which happens to be the third spot in the street.
- The third, and last, car enters the street and parks in spot 1, its preferred spot.

Since all cars were able to park we now know that (2, 2, 1) is a parking function. Let's now consider (3, 2, 3). We find the following:

- The first car enters the street and parks in its preferred spot numbered 3.
- The second car enters the street and goes to the second parking spot (its preferred spot) and finds it available, so it parks there.
- The third, and last, car enters the street and approaches the third parking spot, its preferred spot, but finds that car 1 already parked there! So, it continues down the street, but there are no parking spots after the third spot.

Since the third car cannot park, we know that (3, 2, 3) is not a parking function.

 Working from your list of parking preference vectors test the remaining preference vectors to determine which ones are parking functions. When you are done, check your work against the list provided below and continue on to the next section.

Below is the list of preference vectors which are parking functions.

(1, 1, 1)	(1, 1, 2)	(1, 1, 3)	(1, 2, 1)	(1, 2, 2)	(1, 2, 3)	(1, 3, 1)	(1, 3, 2)
(2, 1, 1)	(2, 1, 2)	(2, 1, 3)	(2, 2, 1)	(2, 3, 1)	(3, 1, 1)	(3, 1, 2)	(3, 2, 1)

Note that there are 16 parking functions when $n = 3$.

2. Counting Parking Functions

In this section, we first present a way to generate all of the parking functions recursively. Then, we share a classical proof by Henry O. Pollak (c. 1974) which accounts for the number of parking functions of length³ n .

2.1 Enumerating Parking Functions Recursively

What does “recursively” mean? “Recursively” describes a method by which we start somewhere, and we do the same thing over and over again to find new things. We can think of it as a special kind of repetition. We often use the word recursion to describe a particular pattern of repetition.

If you have skip counted, you have encountered recursions before! For example, if we were skip counting by 2, and started at 13, that would describe a recursion. Starting at 13, we would add 2 to get 15, and add 2 to get 17, and so on. By recursively doing this, we would find all of the odd numbers greater than 12. The two important parts of a recursion are:

1. The starting point(s) and
2. The rule to generate the next element in a sequence of objects.

If we have a starting point without a rule, we will not be able to generate any other elements of the set. If we define a rule without a starting point, we cannot guarantee that somebody else following the same rule will generate the same set that we generated. Thus, both of these parts are important and necessary in order for us to properly define a recursion.

This article is about parking functions, so let’s think about how we can connect our objects to recursions. Perhaps we could use recursions to help us generate, or list, all possible parking functions of a given length n . That is, if we start with the smallest case, $n = 1$, we can list all of the possible parking functions: (1). That is our starting point. Then, we want to determine a recursion that we can follow in order to generate all of the parking functions for the next case, $n = 2$. We will construct such a recursive rule and use it to prove the following proposition.

In what follows, we let PF_n denote the set of all parking functions of length n , and we let $|PF_n|$ denote the number of parking functions of length n , which is the size of the set PF_n .

Proposition 1. The number of parking functions of length n follows the recursive formula

$$|PF_n| = \sum_{i=1}^n i \binom{n-1}{i-1} |PF_{i-1}| \cdot |PF_{n-i}|. \quad (1)$$

Proof of Proposition 1 as presented in [1]. First, recall that we have n cars and n parking spots. We assume that we have a parking function, meaning that all of the cars are able to park. Note that when it is time for car c_n to park, there is exactly one empty parking spot, which we will say is the i th space. In other words, i represents the number of the first (and only) empty parking spot as car c_n enters the street. We illustrate this in the figure below, and we note that i could be the first or even the last parking spot.

³ Here, we will shorten saying that n is the number of cars and n is the number of parking spots, by simply saying that we are considering parking functions of length n .



Figure 2. Parking spots enumerated for recursion.

Since we assume that car c_n parks in spot i , this implies that all of the parking spots before spot i are occupied; that is, the first $i - 1$ spots, those numbered from 1 to $i - 1$, are occupied.

Let C be the set of cars c_1, \dots, c_{n-1} . We now can create a subset of C consisting of the cars which parked in the parking spots numbered 1 to $i - 1$, inclusive. Let us call this subset of cars the subset S . Since these cars occupied the first $i - 1$ spots, the number of ways they could have preferred these spots is precisely $|PF_{i-1}|$. This then accounts for the factor of $|PF_{i-1}|$ appearing in the right-hand side of equation 1.

Now, we study the remaining cars in C that are not in S ; we call this subset of cars the complement of S , denoted by S' . Note that no car in S' can park in parking spots 1 to $i - 1$ since those spaces are occupied by the cars in the set S . Thus, the cars in S' have preferences that form a parking function on the parking spots numbered $i + 1$ to n because we must have that car c_n parks in spot i . Now, we can count that there are $n - i$ parking spots in that interval, and we know that there are $|PF_{n-i}|$ parking functions on those parking spots. This then accounts for the factor of $|PF_{n-i}|$ appearing in the right-hand side of equation 1.

Observe that we have now split the first $n - 1$ cars into two disjoint subsets, S and S' , where S contains $i - 1$ cars which parked in the first $i - 1$ spaces, while the remaining $n - i$ cars which make up the set S' park in the last $n - i$ spaces. Next, we note that $\binom{n-1}{i-1}$ counts the number of ways in which we can select the $i - 1$ cars from the available $n - 1$ cars in order to create the set S . Once those cars make up set S , then the remaining cars (those not in S) are the cars in the set S' . This then accounts for the factor of $\binom{n-1}{i-1}$ appearing in the right-hand side of equation 1.

Lastly, we consider the preference of car c_n . Recall our parking rule, which states that if a car's preferred spot is taken, the car must drive down the street and park in the first available spot. Thus, we need to account for the fact that car c_n can have a preference of a parking spot numbered anywhere from 1 to i , inclusive. Seeing how the spots 1 through $i - 1$ are occupied by the cars in the set S , this guarantees that car c_n will park in spot i . Observe, there are i possible preferences of car c_n . This then accounts for the factor of i appearing in the right-hand side of equation 1.

Assembling these factors, we determine that the possible parking functions resulting in car c_n parking in spot i is given by the product of these respective factors. Note this holds because each of these choices is independent of the other. More specifically, there are

$$|PF_{i-1}| \cdot |PF_{n-i}| \binom{n-1}{i-1} i$$

possible ways in which car c_n can park in spot i . We end by accounting for the fact that i can be 1 through n , inclusive. Thus, we can sum over all possible values of i to find that the total number of parking functions is

$$|PF_n| = \sum_{i=1}^n i \binom{n-1}{i-1} |PF_{i-1}| \cdot |PF_{n-i}|$$

as we claimed. \square

2.2 A Formula for the Number of Parking Functions

A shortcoming of only having a recursive formula for parking functions is that in order to determine the total number of parking functions of length $n + 1$, we would need to compute the number of parking functions of length 1 through n . As we saw in our previous examples, that can be a lot of work! So, if we are interested in the total number of parking functions, we might not want to first generate them all and then count them. Instead, we might want to have a formula that simply tells us the total number of parking functions. This is precisely what Richard P. Stanley, Alan G. Konheim, and Benjamin Weiss did, and (spoiler alert!) we summarize in the following result.

Theorem 1 (Pyke, 1959; Konheim and Weiss, 1966). The number of parking functions of length n is $|PF_n| = (n + 1)^{n-1}$.

Pollak's proof⁴ happens to be just six sentences, and we state them shortly. However, this level of "conciseness" can make it difficult to understand as it covers quite a bit of mathematical material. So, in Part 2, we will go through it line by line and provide some additional details so that we can make sense of the mathematics together.

Proof of Theorem 1 as presented in [2]. "Add an additional space $n + 1$ and arrange the spaces in a circle. Allow $n + 1$ also as a preferred space. Now all cars can park, and there will be one empty space. α is a parking function if and only if the empty space is $n + 1$. If $\alpha = (a_1, \dots, a_n)$ leads to car C_i parking at space p_1 , then $(a_1 + j, \dots, a_n + j)$ (modulo $n + 1$) will lead to car C_i parking at space $p_i + j$. Hence, exactly one of the vectors $(a_1 + k, a_2 + k, \dots, a_n + k)$ (modulo $n + 1$) is a parking function, so

$$|PF_n| = \frac{(n+1)^n}{n+1} = (n+1)^{n-1}." \quad \square$$

Before we jump into the proof, let us take a moment to explain modular arithmetic. You have encountered this before too because modular arithmetic is very closely related to how we denote time. In modular arithmetic, we only care about integers up to multiples of a fixed whole number called the **modulus**. In other words, if we let m be the modulus, then we consider two integers x and y to be equal "modulo m " if x and y differ by a multiple of m . We write this as " $x = y \pmod{m}$ ". Notice that every integer x is equal, modulo m , to the remainder you get when you divide x by m .

Try to prove that $|PF_n| = (n + 1)^{n-1}$ before Part 2 appears.

References

- [1] N. Shales. Recursively counting a parking function, retrieved October 10, 2020. <https://math.stackexchange.com/questions/2718303/recursively-counting-a-parking-function>.
- [2] R. P. Stanley. Parking functions, 2018, retrieved October 8, 2020. <http://www-math.mit.edu/rstan/transparencies/parking.pdf>.

⁴ Sometimes, it is difficult to cite one's sources. As a matter of interest, Pollak's proof is very hard to find. This proof was written in 1974 and is cited as such in many other mathematical articles, yet the original proof, written by Pollak himself – not just cited in others' work – appears to be lost forever.

The Saga of Fran & Fred



by Kateri Escobar Doran, Violet Freimark, Esmé Krom,
Molly Roughan, and Viola Shephard

THE SAGA OF MATHEMATICAL MATHNESS

Sir Francis III, Esquire, is chasing after his evil twin brother — Sir Frederick III — who is trying to take over the estate and lifetime supply of bananas entrusted to Francis, along with the Knightdom of the Knightly Knights of ABBABA. Unfortunately, his fellow knights believe Fran is the bad guy, due to Fred’s popularity (and excellent banana smoothie recipe, which made him famous among all gorilla-kind). Now Fred is chasing after Francis, knights in tow. Fortunately, Francis has a plan — throw as many banana peels as he can at Fred in the hopes that he slips. Fred will only slip if at least one banana peel sticks to 3 out of his 4 limbs, or if there is at least one under each foot. As a true Mathematical Gorilla Knight of ABBABA, Fran decides to predict the outcome with a proof. What are the chances that Sir Fred will slip and Sir Fran will escape?

Problem Statement

This session at Girls’ Angle, we have been working on a probability problem that involves throwing banana peels at gorillas — a totally normal thing to do.

The basic assumptions of the problem are as follows:

A gorilla has 2 hands and 2 feet — 4 limbs in total. We throw a certain number of banana peels at him, and he slips if at least 1 banana peel lands on each foot, or if banana peels land on at least 3 limbs. The question is, **if we throw n banana peels at the gorilla, what is the probability that he will slip?**

(Note: In this problem, we assume a) that our banana-throwing aim is perfect, so each banana peel we throw will always hit one of the gorilla’s limbs, b) that when a banana lands on a limb, it sticks there, and c) that the bananas we throw are the only thing making him slip and, thus, contributing to the calculated probability of slippage.)

To make this article easier to read, we have come up with the color-coding system for different segments of the problem shown in the box at right.

bananas (n) = orange
limbs (4) = pink
total # ways to throw = red
ways to slip = green
ways not to slip = blue
ways to choose 2 limbs that are not both feet (5) = purple

Process

We started out solving the problem by manually writing out the number of ways for the gorilla to slip for the first few cases — the one in which we throw 2 bananas at the gorilla and the one in which we throw 3. (The 0- and 1-banana cases are easy because the probability of slippage is 0% in these cases, based on the rules of the problem.)

This involved finding the **# ways for the gorilla to slip** and the **total # ways to throw** 2 or 3 bananas, because the probability of the gorilla slipping is:

$$\text{\# ways to slip} \div \text{total \# ways to throw}$$

For example, the **total # ways to throw** 2 bananas at the gorilla is 16 because there are four ways to throw the first banana and 4 ways to throw the second banana and $4 \times 4 = 4^2 = 16$ **ways to throw 2 bananas**.

So the denominator of our probability fraction is 16.

There are only **2 ways for the gorilla to slip** if we throw 2 bananas at him, because he can only slip if the bananas land on both feet — A and B — in either of the following 2 combinations: AB or BA. (He does not slip if both bananas land on the same limb, as in AA or BB.)

Therefore, the finished fraction for the 2-banana case is:

$$2 \div 16 = 1 \div 8 = \mathbf{12.5\%}.$$

By the time we got to the 4-banana case (illustrated below), we realized that it would probably be easier to calculate the **# ways for the gorilla not to slip**, since the direct calculation was getting more and more tedious. We could find the probability by subtracting the **# ways not to slip** from the **total # ways to throw**.

The 4-Banana Case

If we throw 4 banana peels at the gorilla, the **probability of him slipping** is:

$$\mathbf{\# \text{ ways for gorilla to slip}} \div \mathbf{\text{total \# ways to throw 4 banana peels}}.$$

In other words:

$$(\mathbf{\text{total \# ways to throw}} - \mathbf{\# \text{ ways for the gorilla not to slip}}) \div \mathbf{\text{total \# ways to throw}}.$$

To find the **total # ways to throw**, we see that this total is equal to the **number of limbs (4)** to the power of the **number of bananas we're throwing (in this case, also 4)**:

$$4 \times 4 \times 4 \times 4 = 4^4 = \mathbf{256 \text{ ways to throw 4 bananas}}.$$

Therefore, we need to compute:

$$(\mathbf{256} - \mathbf{\# \text{ ways not to slip}}) \div \mathbf{256}.$$

Now all we have to do is find the **# ways not to slip**.

Remember that the gorilla slips if he has at least 1 banana peel on each foot. This means that there are two kinds of ways for the 4 banana peels to land where the gorilla doesn't slip — if they land on 2 limbs (as long as they aren't both feet), or if they all land on one limb. Now we have to calculate the number of ways for the bananas to land in each of these configurations.

There are 4 ways for all the bananas to land on one limb — all on limb P, all on limb Q (which are our two hands), all on limb R, and all on limb S (which are our two feet). So that's a *check* on part one of the subproblems!

However, finding the number of ways for the gorilla not to slip when the bananas are landing on two different limbs is going to be trickier. First we'll have to calculate the # ways to choose two limbs (6 ways minus the one involving both feet, which would result in slippage, so 5 ways), and then we have to multiply that by the number of ways to write 4 A's or B's in a row (for example, 1A and 3B's, 2A's and 2B's, 3A's and 1B, and vice versa), where there is at least one A and one B (to avoid double counting the case where all the bananas land on a single limb).

Originally, we tried to manually write out the combinations. For example, for 1A and 3B's, we would write out:

ABBB
BABB
BBAB
BBBA

etc., which led to a lot of digression about the Knightly Knights of ABBABA. But eventually we came up with a slicker way to calculate these combinations, like so:

There are $2 \times 2 \times 2 \times 2 = 2^4 = 16$ ways to write a string of 4 A's and B's. Two of these only land on one limb (AAAA and BBBB), so there are $16 - 2 = 14$ ways.

Therefore, there are $5 \times 14 = 70$ ways for the gorilla not to slip if the bananas land on two different limbs. Adding this to the 4 ways if the bananas all land on one limb, we get 74 ways for the gorilla not to slip.

Plugging this into our formula, we see that the probability of gorilla slipping if we throw 4 bananas at him is:

$$(256 - 74) \div 256 = 182 \div 256 \approx 72\%.$$

We repeated a similar process for the 5-banana case, by which point we started to notice a pattern in the # ways not to slip:

# bananas	2	3	4	5
# ways not to slip	14	34	74	154

Notice that

$$14 + 20 = 34, 34 + 40 = 74, \text{ and } 74 + 80 = 154.$$

This convenient sequence (+20, +40, +80) doubled every time. It led to our predicting that the # ways not to slip for the 6-banana case is $154 + 160 = 314$. We checked this manually, and, to our relief, we got an affirmative result!

By this point, our ABBABAs were getting increasingly longer and more difficult to work with, so the logical next course of action was to come up with a simple formula that went straight from n (# bananas) to the probability of the gorilla slipping.

To do this, we decided to take a closer look at our pattern for # ways not to slip. We decided to add 6 to our numbers (14, 34, 74, etc.) in order to make them more manageable (20, 40, 80, etc.).

This is what we found:

$$\begin{aligned} 2: 20 - 6 &= 20 \times 2^0 - 6 = 20 \times 2^{2-2} - 6 \\ 3: 40 - 6 &= 20 \times 2^1 - 6 = 20 \times 2^{3-2} - 6 \\ 4: 80 - 6 &= 20 \times 2^2 - 6 = 20 \times 2^{4-2} - 6 \\ 5: 160 - 6 &= 20 \times 2^3 - 6 = 20 \times 2^{5-2} - 6 \\ 6: 320 - 6 &= 20 \times 2^4 - 6 = 20 \times 2^{6-2} - 6 \end{aligned}$$

... etc.

This beautiful discovery was exactly the breakthrough we needed to complete our formula, AKA the Banana Conjecture:

The probability of gorilla slipping when n bananas are thrown at him is equal to

$$(4^n - (20 \times 2^{n-2} - 6)) \div 4^n.$$

Great! So we're pretty much done, right? Now all we had to do was prove the formula.

The n -Banana Case

We decided to prove the Banana Conjecture by generalizing the 4-banana case to the n -banana case. We figured out the total number of ways to slip by finding a way that we could follow every time. The formula went as follows:

$$\text{probability of gorilla slipping} = \frac{(\text{total \# ways to throw } n \text{ bananas} - \text{\# ways not to slip})}{\text{total \# ways to throw } n \text{ bananas}}$$

Total # ways to throw = denominator = 4^n .

There are 2 different ways not to slip — all n on one limb or 2 limbs that are not both feet. There are four ways for all n banana peels to land on one limb. The number of ways to throw the n banana peels so they landed on 2 limbs that were not both feet was

$$(\text{\# ways to choose 2 limbs that are not both feet}) \times (\text{\# ways to combine } n \text{ A's and B's}).$$

There are 5 ways to choose the 2 limbs without picking both feet (if our limbs are P, Q, R, and S, where R and S are the feet, then the 5 ways are PQ, PR, PS, QR, and QS). Then, after we have chosen the two limbs (A and B) there are $2^n - 2$ ways to throw n banana peels so that they land on these two limbs.

Then we would take our denominator (4^n) and then take our number which we just got and then subtract that from the denominator. That would get you the numerator. Then you could put the two numbers together and you've got the n banana case (our answer):

$$(4^n - (5 \times (2^n - 2) + 4)) \div 4^n = (4^n - (20 \times 2^{n-2} - 6)) \div 4^n.$$

Conclusion

We started by trying to figure out the probability of Sir Fred slipping, and while at first it was easiest to manually count out the number of ways he could slip — or, as we were increasingly beginning to find, not slip — we soon realized that we could come up with a proof through conjecture. This formula will work for any value of n (AKA # bananas):

The probability of gorilla slipping if we throw n bananas at him is equal to

$$(4^n - (20 \times 2^{n-2} - 6)) \div 4^n,$$

or, depending on preference,

$$(4^n - (5 \times (2^n - 2) + 4)) \div 4^n.$$

Our sequence for # ways not to slip (14, 34, 74, 154, 314, etc.) is not currently in the On-Line Encyclopedia of Integer Sequences, which means we are probably the first to discover it — almost certainly the first to discover it in this manner! We are all so excited to have gone on this journey together with Fran and Fred.

However, the adventure is not over yet. We are curious about how our functions and sequence would be altered if we were to add additional rules — what would happen if the gorilla would also slip with, say, 93 banana peels on one limb? — and we may take this project in a new direction in the future!

THE SAGA OF MATHEMATICAL MATHNESS (continued and concluded)

The battle was fierce. Bananas flew everywhere. Afterwards, when the forest was littered with banana peels and Sir Fred had slipped a great many, many times, the two estranged brothers looked around and realized that with an infinite supply of bananas, the only thing to do was make an infinite supply of banana smoothies.

Sir Fred shared his banana smoothie recipe with the Knightly Knights of ABBABA, and Sir Fran decided to open his castle to all the gorilla-knights of the land for the celebration. But the smoothie bar had only just been opened and the first of the guests were still arriving when the castle gates swung open and a mysterious hooded figure came riding in on a bananicorn...

It was their long-lost older sister and the TRUE heir to the knightdom, Lady Francesca Frannifred the First, came to claim her birthright! (And to drink banana smoothies. Adventuring travelling bananicorns are very fond of smoothies.) Sir Fran gladly handed the castle over to Francesca, who decided to share the banana smoothie recipe throughout the land. The castle became a twenty-four hour smoothie bar, and Francis, Fred, and Francesca became best friends. They lived forever in the smoothie bar, drinking banana smoothies to their heart's content.

The End!!!

You Do Math

by Girls' Angle Staff

In her Girls' Angle interview,¹ Prof. Elizabeth Meckes said, "... learning math and doing math are really the same thing; it's an active process. There's a reason that math teachers assign so much homework: there's a real limit to how much understanding you can gain by watching someone else do math. To really understand something you have to work through it yourself. This is not in any way to suggest that teachers can't help; the insights that math teachers can provide about how to think about things can be incredibly valuable, but only in conjunction with struggling with it on your own. And I do mean struggling; understanding math doesn't come easy to anyone (if it has to you so far, don't worry—if you keep at it long enough you'll get to be as confused as the best of them)."

Doing math is a process of asking math questions and trying to answer them. A math question is a question that can be addressed without appeal to anything but logic. It takes practice to come up with good math questions. A good math question enables you to further your understanding. Typically, this means that you feel that you have some ideas, even if vague, for what to try to do to answer the question, and that you can start writing on a piece of scratch paper.

If you find it difficult to come up with a math question, try to actively read math. Inevitably, you will read something that causes you to be confused and makes you wonder about something. When you are in that state, articulating what you are confused about will bring you to the verge of asking math questions.

For example, consider the *Toblerone Game* on page 11. To prove their theorem, the authors had to internalize the logic involved and their proof is written from the vantage point of having understood. So if you have not thought about such things before, when you read their proof carefully, you will likely wonder about how, exactly, does their induction step prove their theorem? After all, it isn't a straightforward induction where you can repeatedly apply the same inductive step to get to any specific case of the theorem, starting from the base case. Perhaps you might even ask if you can come up with a proof that does not use induction? Or, does their game make you think of a game to analyze? Perhaps you have a favorite chocolate bar that is different from the Toblerone bar. What game can you invent for sharing your favorite bar?

Or, consider *The Saga of Fran and Fred* on page 22. How does their answer change if the condition for when a gorilla slips changes? Or, what slippage conditions could you devise for a creature with more than 4 limbs, such as a spider? Is there a general class of problems into which you can fit all animal slippage problems?

What questions does *Parking Functions*, on page 15, make you think of? (If you get stuck here, the authors explicitly address this in the upcoming Part 2.)

We often see people coming up with great questions, but then they dismiss them. Thinking takes energy, but thinking also yields many benefits. So when a question occurs to you, don't drop it. Record it in a math journal and give it some thought. Before long, you'll be asking and answering math questions that nobody has asked or been able to answer before!

¹ See Volume 4, Numbers 2 and 3 of this Bulletin.

Calendar

Session 27: (all dates in 2020)

September	10	Start of the twenty-seventh session!
	17	
	24	
October	1	
	8	
	15	
	22	
	29	
November	5	
	12	
	19	
	26	Thanksgiving - No meet
December	3	

Session 28: (all dates in 2021)

January	28	Start of the twenty-eighth session!
February	4	
	11	
	18	No meet
	25	
March	4	
	11	
	18	
	25	No meet
April	1	
	8	
	15	
	22	No meet
	29	
May	6	

Girls' Angle has run over 150 Math Collaborations. Math Collaborations are fun, fully collaborative, math events that can be adapted to a variety of group sizes and skill levels. We now have versions where all can participate remotely. We have run two such "all-virtual" Math Collaborations, one at the Buckingham, Browne, and Nichols Middle School and one for the PROMYS Girls' Math Circle. If interested, contact us at girlsangle@gmail.com. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

Girls' Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) _____ (first) _____

Parents/Guardians: _____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: _____ Cell Phone: _____

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

The \$50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- Enclosed is a check for \$50 for a 1-year Girls' Angle Membership.
- I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.



A Math Club for Girls

Girls' Angle Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is \$20/meet for members and \$30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle is based in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, founder and director of the Exploratory
Yaim Cooper, Institute for Advanced Study
Julia Elisenda Grigsby, professor of mathematics, Boston College
Kay Kirkpatrick, associate professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, assistant dean and director teaching & learning, Stanford University
Lauren McGough, postdoctoral fellow, University of Chicago
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, associate professor, University of Utah School of Medicine
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Liz Simon, graduate student, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, associate professor, University of Washington
Karen Willcox, Director, Oden Institute for Computational Engineering and Sciences, UT Austin
Lauren Williams, professor of mathematics, Harvard University

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) _____ (first) _____

Parents/Guardians: _____

Address: _____ Zip Code: _____

Home Phone: _____ Cell Phone: _____ Email: _____

Please fill out the information in this box.

Emergency contact name and number: _____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

(Parent/Guardian Signature) Date: _____

Participant Signature: _____

Members: Please choose one.

- Enclosed is \$216 for one session (12 meets)
- I will pay on a per meet basis at \$20/meet.

Nonmembers: Please choose one.

- I will pay on a per meet basis at \$30/meet.
- I'm including \$50 to become a member, and I have selected an item from the left.

I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

**Girls' Angle: A Math Club for Girls
Liability Waiver**

I, the undersigned parent or guardian of the following minor(s)

_____ ,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: _____ Date: _____

Print name of applicant/parent: _____

Print name(s) of child(ren) in program: _____