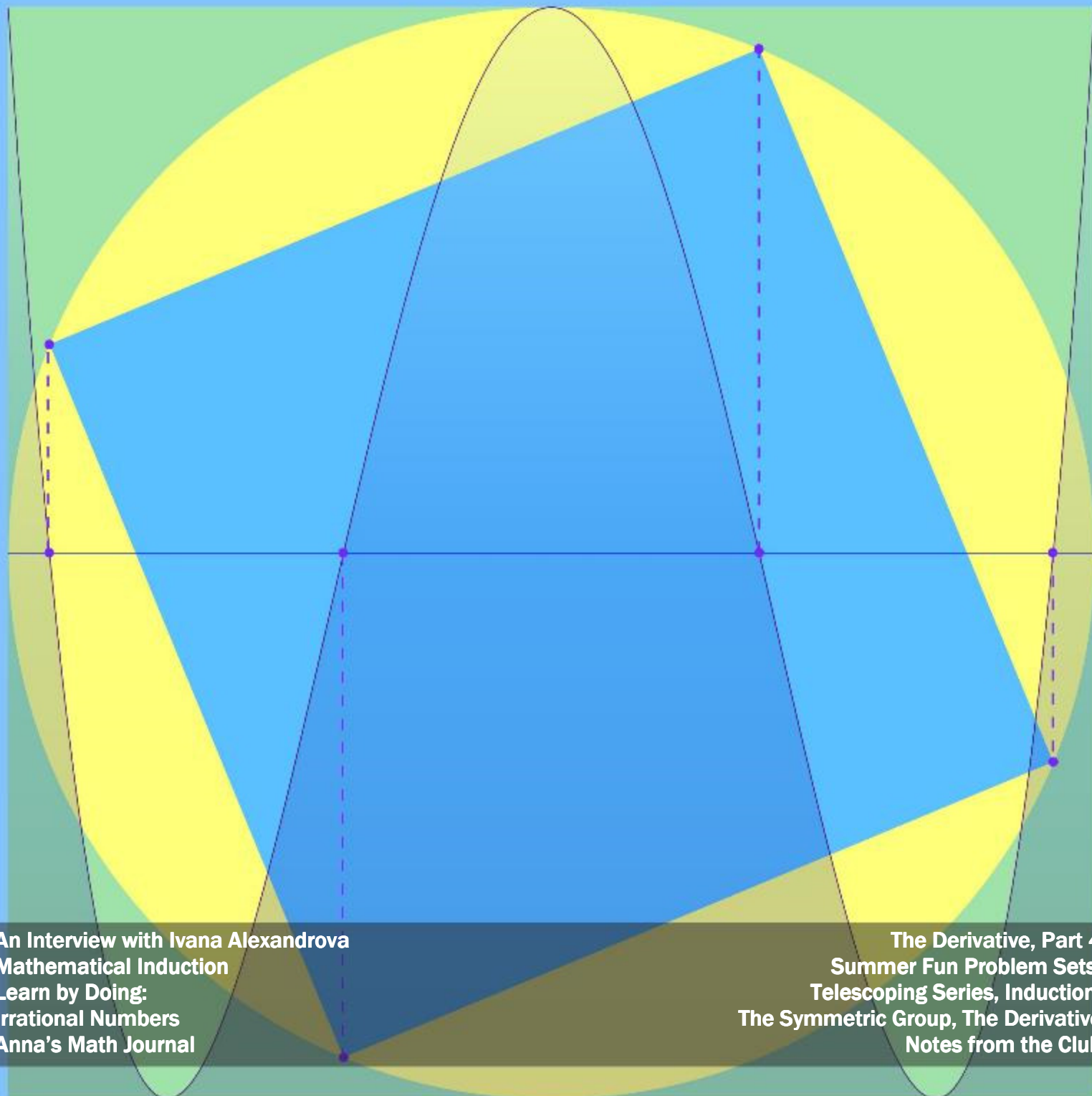


# Girls' *Angle* Bulletin

June/July 2015 • Volume 8 • Number 5

*To Foster and Nurture Girls' Interest in Mathematics*



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Mathematical Induction  
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## From the Founder

Have you ever found learning something new easier because you were able to apply a skill you had learned long ago while learning something completely different? For example, learning about the musical overtone series when I was a child helped me to understand Fourier analysis years later. Don't worry so much about what something's immediately good for. If it makes you think, it's worth learning.

- Ken Fan, President and Founder

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## Girls' Angle Bulletin

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Girls' Angle welcomes submissions that pertain to mathematics.

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## Girls' Angle: A Math Club for Girls

*The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.*

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On the cover:  $T_4$  by C. Kenneth Fan. The fourth Chebyshev polynomial has roots at the vertical projections of the corners of the square. As the square rotates, the polynomial can be shifted vertically to stay so aligned.

# An Interview with Ivana Alexandrova

Ivana Alexandrova was born and raised in Bulgaria where she was accepted into and attended the math high school in her hometown of Vratsa. Throughout middle school and high school she participated in math olympiads and competitions, winning third place in the prestigious Spring National Math Competition in Bulgaria as a ninth grader. She later went on to earn her Ph.D. in mathematics from the University of California, Berkeley. Currently, she is an assistant professor of mathematics at SUNY Albany, where, in addition to her regular research and teaching duties, she maintains a webpage with weekly problems for high school students.

**Ken:** What's the most exciting mathematical concept you recall learning before college? Why do you find it exciting?

**Ivana:** In high school I really liked trigonometry. I liked everything about it but in particular, I liked that trigonometric functions had all these very interesting properties and satisfied all these interesting identities and I liked how powerful that made them – one can use trigonometric functions to solve problems that have nothing a priori to do with trigonometry but whose solutions would be long, difficult, and tedious without introducing the trigonometric functions to solve them. One example is given in the problem for Week 26 of my problems page for high school students in which certain solutions to a seventh degree polynomial are requested. The problem seemingly uses no trigonometry, but it is simplified significantly by the use of a suitable trigonometric substitution.

**Ken:** Could you describe your research area?



Ivana Alexandrova

*...your high school years are a great time to develop strong problems solving skills by working on harder problems than what might be in regular homework sets. To become a research mathematician, one would still need to develop other skills ... but strong problem solving skills can be very helpful in and outside of mathematics. I know many people with strong problem solving skills ... who have applied them with great success in other areas of life.*

The Week 26 problem from Ivana's website asks readers to solve the equation

$$8x(2x^2 - 1)(8x^4 - 8x^2 + 1) = 1$$

for  $x \in (0, 1)$ .

For more, visit:

[www.albany.edu/ialexandrova/HS.html](http://www.albany.edu/ialexandrova/HS.html)

**Ivana:** I work in a subarea of partial differential equations called scattering theory. It is motivated by the physical sciences. Scattering theory studies interacting physical systems on scales of time and/or distance much larger than those of the interaction region itself. Scattering experiments are used throughout the sciences to study objects which are far away (like the stars), very small (like the atom), or difficult to reach (like the Earth's core). In a typical scattering experiment a signal is sent towards the object and is compared to the one scattered back with the goal of recovering information about the structure of the object (or the scatterer). Similar processes are employed, for example, in the studies of the chemical composition of the stars or in determining where to drill for oil.

**Ken:** Can you explain one of your own results? How did you discover or create it?

**Ivana:** Maybe I can tell you briefly about my last paper. In it I study the finer structure of the scattering amplitude in the presence of a magnetic field. The scattering amplitude is roughly the operator, which in scattering experiments assigns the outgoing data to the incoming ones. Understanding its structure in various settings is often crucial to being able to recover information about the scatterer. The structure of the scattering amplitude I was able to uncover in this case was very new and unexpected. Also, to determine and completely characterize this structure I had to create a new class of operators, which was quite challenging because it had to be a class of operators which both made sense within the already existing framework and helped me solve my problem. It felt very rewarding when I was able to accomplish all that.

**Ken:** In your list of publications, I saw some titles about the Aharonov-Bohm effect. What is the Aharonov-Bohm effect?

**Ivana:** The Aharonov-Bohm effect was discovered in the 1950s by physicists Yakir Aharonov and David Bohm. In broad strokes, the Aharonov-Bohm effect is a quantum-mechanical phenomenon in which an electrically charged particle shows a measurable interaction with a physical system even in regions of space where the force field is zero. Prior to its discovery and ever since Newton wrote his famous equations of motion in the 17<sup>th</sup> century, physical theory had been formulated in terms of forces and force fields (e.g., magnetic fields). The Aharonov-Bohm effect, however, suggests that not forces and force fields but rather potential energy should be used as the starting point for formulating physical theory. This constitutes a major revision of physical principles. For that reason the Aharonov-Bohm effect has been named one of the “seven wonders of the quantum world” by The New Scientist magazine. Some of my research, joint with Hideo Tamura from Okayama University in Japan, has studied manifestations of the Aharonov-Bohm effect in scattering theory.

**Ken:** Do you consider yourself a physicist or a mathematician? Is there a distinction? Does it matter?

**Ivana:** I absolutely am a very pure mathematician! :-) I work in areas of research where the problems are motivated by physical phenomena but I work on the math side of things where I prove theorems to the last epsilon. Nothing is handwavy about what I do. In my not so extensive experience talking with physicists and reading their papers, I have observed many differences between the way physicists and mathematicians approach scientific questions. In most physics papers, for example, there are no theorems or justifications of results, there is just a free flowing discussion. You cannot do that in pure math papers and I do not do any of that. With all this I am not saying that I do not

respect what physicists do – I very much do respect their work – I am just not a physicist and I could never be one because apparently my brain can process only very precise and rigorous thoughts. But, as they say, to each her own. :-)

**Ken:** What do you enjoy about being a mathematician? What is your life like as a mathematician?

**Ivana:** I enjoy the mental challenge of solving a difficult problem as well as the element of surprise that comes from uncovering new and unexpected truths. I also enjoy the unexpected scientific encounters with other researchers I meet along my scientific journey. My daily life is not that different from ordinary daily lives but I do set aside some time on a regular basis for sinking deeply into some mathematical question. That feels like meditation and like something that resets my life goals in a more meaningful direction.

**Ken:** What characteristics do you have that have helped you succeed as a mathematician?

**Ivana:** Good problem solving skills help but also for me it has always been important to try to get to the root of the problem, to understand the underlying reason why things work (or don't work). Being able to communicate with and learn from mathematicians in other areas, on the other hand, has helped me take my research in new and exciting directions. Being able to explain my ideas clearly to as wide an audience as possible, both orally and in writing, has also been important to the growth of my career.

**Ken:** You post a weekly problem for high school students on your website. What are your goals for that and what is your intended audience?

**Ivana:** Thank you for asking me this question. Yes, at the beginning of this year I created the website

[www.albany.edu/ialexandrova/HS.html](http://www.albany.edu/ialexandrova/HS.html)

where I post weekly math problems for high school students. I created this website because after many years of teaching mathematics at the college level in the US I have observed the regrettable reality that US educated students are not as well prepared for college level math courses as their international counterparts and thus have a harder time doing well in math in college. I do not believe that this is the fault of the students themselves – they do work and try to do the homework and are not in any way genetically inferior to the international students I have had in my classes – I believe that the system here simply does not expose students to enough or the right material to prepare them better for college. That gave me the idea to start posting problems on my website which I believe will help high school students prepare better for college level mathematics. Of course, they need to do many more practice problems than what I am able to post on my website but I thought I would provide some good problems for them to think about. The website is for any high school student who wants to learn more math and be better prepared for college. I am especially hoping to reach students who, due to their socioeconomic circumstances, may not have access to the best college preparation courses, guidance counselors and mentors. That is why the content on my website is absolutely free and accessible to anyone even with the slowest Internet connections and there are no ads on my website.

**Ken:** What are some sample problems from your website? What was your aim in choosing them?



*I do set aside some time on a regular basis for sinking deeply into some mathematical question. That feels like meditation and like something that resets my life goals in a more meaningful direction.*

**Ivana:** One of the problems is to solve the inequality

$$\left| |x+1| - |x-3| \right| > 5 - |x+2|.$$

I chose this problem because in my experience too many college students do not feel comfortable working with absolute values, especially when there are several of them and/or are nested. I wanted to show the students that if they break up the problem into a few smaller logical steps, it becomes manageable.

Another problem from my website asks the students to evaluate the expression

$$\cos 36^\circ \cdot \cos 72^\circ.$$

I chose this problem, as well as all the other trigonometry problems, because again I see too many college students who do not have a good enough mastery of trigonometry and so I am trying to provide some interesting trigonometry problems for them to learn from.

I also like problems that involve different branches of mathematics because it is important to be able to see the connections among the different branches of mathematics. For example, the problem for Week 12 was to determine whether there exist real numbers  $x$  such that

$$b^2x^2 + (b^2 + c^2 - a^2)x + c^2 = 0,$$

where  $a$ ,  $b$ , and  $c$  are the lengths of the sides of a triangle. This problem combines geometry and algebra.

**Ken:** What advice do you have for a teen who aspires to become a mathematician? What should she study? How should she allot her “math time”?

**Ivana:** There are different things that students at that age do and different things work for different people – some take advanced college level and even graduate level courses, some participate in math circles and math contests – so every teen should make the decision of how to allot her math time based on her own interests and on the advice she may receive from her mentors, teachers, and parents. I would say, however, that taking college and graduate level courses as a teen is rare, even among the most successful mathematicians, and that is for a reason. I do believe, on the other hand, that your high school years are a great time to develop strong problems solving skills by working on harder problems than what might be in regular homework sets. To become a research mathematician, one would still need to develop other skills in college and graduate school but strong problem solving skills can be very helpful in and outside of mathematics. I know many people with strong problem solving skills, which they have developed exactly through working on math problems and participating in math contests, who have applied them with great success in other areas of life. So, while every teen should filter all advice, including this one, through their own experience and situation, I would just like them to know that strong problem solving skills can be a very powerful asset in life. And I will be posting some good problems on my website on which to hone those skills.

**Ken:** Thank you for this interview!

# Mathematical Induction

by Ken Fan | edited by Jennifer Silva

To master the technique of induction, practice it. Try your hand at the Summer Fun problem set on page 22.

Mathematical induction is an important and commonly used proof technique.

Dear Reader,

We're committed to producing quality math educational content and make every effort to provide this content to you for free.

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For this issue, those who do not subscribe to the print version will be missing out on this expository article on induction. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

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Thank you and best wishes,  
Ken Fan  
President and Founder  
Girls' Angle: A Math Club for Girls

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# Learn by Doing

## Irrational Numbers

by Girls' Angle Staff

Supposedly (although who knows?) the ancient Greeks, for a very long time, thought that if you took a measurement of length, you would always get a **rational number**, which is a number that can be expressed as the ratio of two integers, like  $1/3$ ,  $-3/2$ , or  $22/7$ .

**Problem 1.** Consider a square one unit on a side. Use the Pythagorean theorem to determine the length of either of its diagonals.

**Problem 2.** Early ancient Greeks supposed that  $\sqrt{2}$ , which is the answer to Problem 1, was some rational number. So it was natural for them to ask: what integers  $p$  and  $q$  satisfy

$$\sqrt{2} = \frac{p}{q}.$$

Is it  $7/5$ ? Well,  $(7/5)^2 = 49/25$ , which is not quite 2. How about  $17/12$ ? If we square  $17/12$ , we get  $289/144$ , which differs from 2 by only  $1/144$ . But, that's still not equal to 2.

Without loss of generality, we may assume that  $p$  and  $q$  share no common factor greater than 1 (by writing the fraction in lowest terms). The equation  $\sqrt{2} = p/q$  can be rearranged to  $\sqrt{2} q = p$ . Squaring both sides yields  $2q^2 = p^2$ . Show that this equation implies that both  $p$  and  $q$  must be even, contradicting the assumption that  $p$  and  $q$  share no common factor greater than 1. Hence, no such  $p$  and  $q$  exist!

Problem 2 shows that there are lengths which are not rational numbers. So we define **irrational numbers** to be those numbers which are not rational. Hence,  $\sqrt{2}$  is an irrational number. Because irrational numbers are defined in terms of what they are not, many proofs that show that a number is irrational are proofs by contradiction. They begin by assuming that the number of interest is rational then deduce a contradiction.

Note: Sometimes we'll write " $x$  is rational" or " $x$  is irrational" to mean that  $x$  is a rational number or  $x$  is an irrational number, respectively.

**Problem 3.** Generalize the argument in Problem 2 to show that  $\sqrt{n}$  is irrational for any integer  $n$  that is not a perfect square.

**Problem 4.** Suppose that  $x$  and  $y$  are numbers such that  $x/y$  is rational. Show that there exists  $z$  such that  $x$  and  $y$  are both integer multiples of  $z$ .

**Problem 5.** Suppose that  $x$  and  $y$  are numbers such that  $x/y$  is irrational. Show that there does not exist  $z$  such that  $x$  and  $y$  are integer multiples of  $z$ .

**Problem 6.** Let  $a$  be an irrational number and let  $r$  be a rational number. Show that  $r + a$  is irrational. Show that  $ra$  is also irrational, provided that  $r \neq 0$ .



**Problem 7.** Find irrational numbers  $a$  and  $b$  whose sum  $a + b$  is rational.

**Problem 8.** Find irrational numbers  $a$  and  $b$  whose product  $ab$  is rational.

**Problem 9.** Show that  $\sqrt{2} + \sqrt{3}$  is irrational.

**Problem 10.** Can you find irrational numbers  $a$  and  $b$  such that  $a^b$  is rational?

**Problem 11.** Show that  $\log_{10} 2$  is irrational.

**Problem 12.** Find at least 4 pairs of positive integers  $b$  and  $n$  such that  $\log_b n$  is irrational.

**Problem 13.** Prove that a number is rational if and only if it has a repeating decimal expansion. (Here, we think of terminating decimals as expansions that have a repeating 0.)

**Problem 14.** Let  $a = \sum_{n=1}^{\infty} 10^{-n(n+1)/2}$ . Prove that  $a$  is irrational. Hint: Use Problem 13.

**Problem 15.** (“rational root theorem”) Let  $p$  and  $q$  be relatively prime integers with  $q \neq 0$ . Suppose  $p/q$  is a root of the equation  $a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0 = 0$ , where all the coefficients  $a_k$  are integers with  $a_0, a_n \neq 0$ . Show that  $p$  must divide  $a_0$  and  $q$  must divide  $a_n$ .

**Problem 16.** Use the rational root theorem to show that  $\sqrt[n]{m}$  is irrational for positive integers  $n$  and  $m$  if and only if  $m$  is not a perfect  $n$ th power.

**Problem 17.** Show that  $\cos(2\pi/5)$  is irrational. Hint: Express  $\cos(5x)$  as a polynomial in  $\cos x$ . Then use the rational root theorem.

In Problems 18-21, you’ll reconstruct a proof that  $e$  is irrational attributed to Joseph Fourier. For these problems, we’ll assume that

$$(\odot) \quad e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$$

**Problem 18.** Using  $\odot$ , show that  $2 < e < 3$ .

**Problem 19.** Suppose that  $e = p/q$ , where  $p$  and  $q$  are positive integers. Note that  $q > 1$ . Show that

$$q!e = q! \sum_{n=0}^{\infty} \frac{1}{n!} = q! + \frac{q!}{1!} + \frac{q!}{2!} + \frac{q!}{3!} + \frac{q!}{4!} + \dots = N + \frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \frac{1}{(q+1)(q+2)(q+3)} + \dots,$$

where  $N$  is an integer.

**Problem 20.** Show that  $\frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \frac{1}{(q+1)(q+2)(q+3)} + \dots < \frac{1}{q} < 1$ .



**Problem 21.** Prove that  $e$  is irrational.

In Problems 22-28, you'll reconstruct a proof of Charles Hermite<sup>1</sup> that  $\pi$  is irrational. You'll need to be familiar with integration and know how to integrate by parts to complete these problems.

Following Hermite, define  $A_n = \frac{x^{2n+1}}{2^n n!} \int_0^1 (1-z^2)^n \cos(xz) dz$ .

**Problem 22.** Show that  $A_0 = \sin x$ . (Note: by convention,  $0! = 1$ .)

**Problem 23.** Show that  $A_1 = \sin x - x \cos x$ .

**Problem 24.** Show that  $A_n = (2n-1)A_{n-1} - x^2 A_{n-2}$  for  $n > 1$ . Hint: Use integration by parts twice. For another hint, start by applying integration by parts once to show that

$$A_n = \frac{x^{2n}}{2^{n-1}(n-1)!} \int_0^1 z(1-z^2)^{n-1} \sin(xz) dz.$$

Then apply integration by parts again to this integral.

**Problem 25.** Use induction to show that  $A_n = U_n(x)\sin x + V_n(x)\cos x$ , where  $U_n(x)$  is a polynomial in  $x^2$  with integer coefficients and degree  $n/2$  or  $(n-1)/2$  depending on whether  $n$  is even or odd, respectively.

**Problem 26.** Show that  $A_n > 0$  if  $x = \pi/2$ .

**Problem 27.** Suppose that  $\pi^2/4 = p/q$  where  $p$  and  $q$  are relatively prime integers. Set  $x = \pi/2$ .

A. For even  $n$ , show that  $U_n(\pi/2) = N_n/q^{n/2}$  where  $N_n$  is an integer. Thus, (for even  $n$  and  $x = \pi/2$ )

$$A_n = \frac{N_n}{q^{n/2}} = \frac{p^{n+1/2}}{2^n n! q^{n+1/2}} \int_0^1 (1-z^2)^n \cos\left(\frac{\pi}{2} z\right) dz.$$

B. Therefore,  $N_n = \frac{p^{n+1/2}}{2^n n! q^{(n+1)/2}} \int_0^1 (1-z^2)^n \cos\left(\frac{\pi}{2} z\right) dz$ . Show that the right-hand side of this equation tends to 0 with  $n$ , whereas the left-hand side is always a nonzero integer. (Hint: the integral is between 0 and 1 because it represents an area contained within a unit square.)

**Problem 28.** Complete the proof that  $\pi$  is irrational.

For more on irrational numbers, we suggest two books by Ivan Niven: *Numbers: Rational and Irrational* and *Irrational Numbers*.

<sup>1</sup> Sur quelques approximations algébriques, *Journal de Crelle* **76** (1873), pp. 342-344.



# Anna's Math Journal

By Anna B.

*Mathematics is a journey of discovery. As mathematicians take this journey, they follow many wrong turns, believe many incorrect facts, and encounter many mysteries. Out of these twists and turns comes the reward of truth and understanding. However, if you look at math books, you might get the impression that mathematicians rarely err. In this column, Anna gives us a peek into her mathematical process of discovery, bravely allowing us to watch even as she stumbles.*

Anna tackles Ivana Alexandrova's 2015 Week 11 problem: Evaluate  $\cos 36^\circ \cos 72^\circ$ .

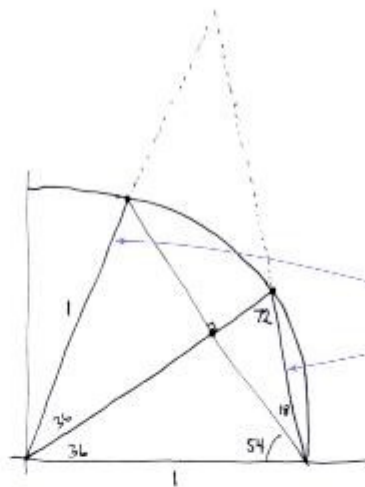
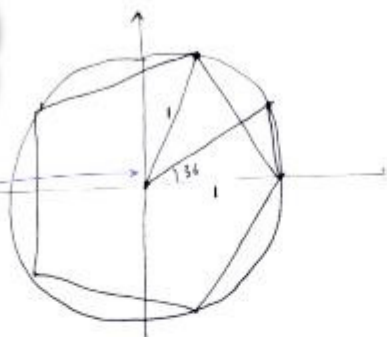
Let's see. The problem is to compute  $\cos 72$  times  $\cos 36$ . Neither angle seems to relate to angles whose cosines I know, like  $30$ ,  $45$ ,  $60$ , or  $90$  degrees. But  $72$  is twice  $36$ , so I know how to compute one from the other by using double or half angle formulas.

$72$  is one-fifth of  $360$ , so  $72$  is a sector angle of a regular pentagon.

I'll try to redraw so that all the isosceles triangles look isosceles.

These two lengths must be the same since they are both one less than the length of the equal sides of the isosceles triangle. I'll label this length  $x$ .

Evaluate  $\cos 72^\circ \cos 36^\circ$



Wait a sec. An isosceles triangle with apex angle  $36$  degrees has a base angle of  $(180 - 36)/2$  or  $72$  degrees. That means if I extend these sides, I should get an isosceles triangle. Whoa... my drawing looks way off!

Hm. Maybe I can apply the law of cosines to the two isosceles triangles with apex angle  $36$  degrees to get 2 equations in  $x$  and  $\cos 36$ , then solve for both.

Applying the law of cosines to the smaller triangle yields...

And to the larger triangle...

Both involve  $2(1 - \cos 36)$ , so I can eliminate that to get an equation involving only  $x$ . Oh...this just expresses that the two isosceles triangles are similar! I should have seen that.

In any case, I can solve for  $x$  using the quadratic formula.

$$x^2 = 1^2 + 1^2 - 2(1)(1) \cos 36$$

$$= 2 - 2 \cos 36$$

$$1^2 = (1+x)^2 + (1+x)^2 - 2(1+x)(1+x) \cos 36$$

$$= 2(1+x)^2 (1 - \cos 36)$$

$$x^2 = \frac{1}{(1+x)^2} \quad x(1+x) = 1$$

$$x^2 + x - 1 = 0$$

$$x = \frac{-1 + \sqrt{5}}{2}$$

Key:

Anna's thoughts

Anna's afterthoughts

Editor's comments



Now I can solve for  $\cos 36^\circ$ .

$$\cos 36^\circ = \frac{2-x^2}{2} = 1 - \frac{x^2}{2} = 1 + \frac{(x-1)}{2} = \frac{1+x}{2} = \frac{1+\sqrt{5}}{4}$$

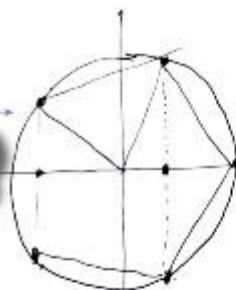
I'll find  $\cos 72^\circ$  using the double angle formula.

$$\cos 72^\circ = \cos^2 36^\circ - \sin^2 36^\circ = 2\cos^2 36^\circ - 1 = 2\left(\frac{1+\sqrt{5}}{4}\right)^2 - 1 = 2\left(\frac{6+2\sqrt{5}}{16}\right) - 1 = \frac{3+\sqrt{5}}{4} - 1 = \frac{\sqrt{5}-1}{4}$$

Now I can evaluate the original expression.

$$\cos 72^\circ \cos 36^\circ = \frac{\sqrt{5}-1}{4} \cdot \frac{\sqrt{5}+1}{4} = \frac{1}{4}$$

$1/4$  is a nice number. I wonder if there's a way to see the answer using a regular pentagon instead of a 36-72-72 isosceles triangle.



$$\cos 36^\circ \cos 72^\circ = -\cos 144^\circ \cos 72^\circ$$

$$(\cos 36^\circ \cos 72^\circ)^2 = \cos 72^\circ \cos 144^\circ \cos 216^\circ \cos 288^\circ$$

$$\cos 5\theta = 1$$

I'll use Euler's formula to figure out what  $\cos 5\theta$  is in terms of  $\cos \theta$ .

$$\begin{aligned} \cos 5\theta + i\sin 5\theta &= (\cos \theta + i\sin \theta)^5 \\ &= \cos^5 \theta + 5\cos^4 \theta (i\sin \theta) + 10\cos^3 \theta (i\sin \theta)^2 + 10\cos^2 \theta (i\sin \theta)^3 + 5\cos \theta (i\sin \theta)^4 + (i\sin \theta)^5 \end{aligned}$$

Equating real parts...

$$\begin{aligned} \cos 5\theta &= \cos^5 \theta - 10\cos^3 \theta \sin^2 \theta + 5\cos \theta \sin^4 \theta \\ &= \cos^5 \theta - 10\cos^3 \theta (1-\cos^2 \theta) + 5\cos \theta (1-\cos^2 \theta)^2 \\ &= 16\cos^5 \theta - 20\cos^3 \theta + 5\cos \theta \end{aligned}$$

If I set  $\cos 5\theta = 1$  and substitute  $x = \cos \theta$ , I'll get a polynomial whose roots are the cosines of the angles in a regular pentagon.

$$\begin{aligned} 16x^5 - 20x^3 + 5x - 1 &= 0 \\ x^5 - \frac{5}{4}x^3 + \frac{5}{16}x - \frac{1}{16} &= 0 \end{aligned}$$

So  $\cos 36^\circ \cos 72^\circ$  must be the square root of  $1/16$ , and it's the positive square root because both  $\cos 36^\circ$  and  $\cos 72^\circ$  are positive.

$$\therefore \cos 0^\circ \cos 72^\circ \cos 144^\circ \cos 216^\circ \cos 288^\circ = \frac{1}{16}$$

$$\Rightarrow \cos 36^\circ \cos 72^\circ = +\frac{1}{4}$$

$$\cos 36^\circ - \cos 72^\circ = \frac{1}{2}$$

$$\cos 72^\circ = 2\cos^2 36^\circ - 1$$

$$\cos 36^\circ + 1 - 2\cos^2 36^\circ = \frac{1}{2}$$

$$4\cos^2 36^\circ - 2\cos 36^\circ - 1 = 0$$

There are a lot of fun patterns to explore in this problem. For more problems like this, visit Ivana Alexandrova's webpage.

Actually,  $\cos 36^\circ = \cos 144^\circ$ . And also  $\cos 72^\circ = \cos 288^\circ$  and  $\cos 144^\circ = \cos 216^\circ$ . And  $\cos 0^\circ = 1$ , so I can express  $\cos 36^\circ \cos 72^\circ$  as the square root of the product of the cosines of all the angles in the regular pentagon.

Since all the angles in a regular pentagon satisfy  $\cos 5\theta = 1$  and  $\cos 5\theta$  is a polynomial in  $\cos \theta$ , I can use Vieta's formulas to get the product of these cosines.

The product of the roots is the negative of the constant term.

Funny that the coefficient of  $x^4$  is 0... oh wait... of course it's 0... the sum of the fifth roots of unity is 0, and this coefficient is the real part. That means  $\cos 36^\circ - \cos 72^\circ = 1/2$ . This fact can be seen just by symmetry, so starting with this, and then using the double angle formula for  $\cos 72^\circ$ , we get an equation for  $\cos 36^\circ$ . That's pretty efficient.

Key:

Anna's thoughts

Anna's afterthoughts

Editor's comments

ABB 6.26.15

# The Derivative, Part 4

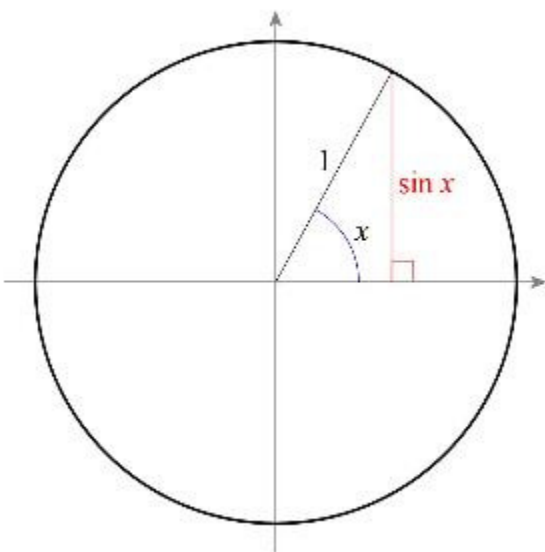
by Ken Fan | edited by Jennifer Silva

In this installment, we find the derivatives of the basic trigonometric functions.

Calculus is a subject well covered in textbooks. Instead of giving another textbook treatment of the subject, we aim to provide an illuminating, though non-rigorous, explanation.

## The derivative of the sine function

In keeping with the spirit of this series on the derivative, we will deduce the derivative of the sine function non-rigorously. Rigorous treatments can be found in numerous texts, such as *Introduction to Calculus and Analysis, Volume I*, by Courant and John.



Let's recall what  $\sin x$  is.

Shown at left is a unit circle centered at the origin of a Cartesian plane. A radial line is drawn that makes an angle of  $x$  radians with the positive horizontal axis, as measured counterclockwise about the origin. This radial line intersects the unit circle, and the vertical coordinate of this point of intersection is the sine of  $x$ . (The cosine of  $x$  is the point's horizontal coordinate.)

As  $x$  increases from 0 to  $2\pi$ , the value of  $\sin x$  goes from 0 to 1, back down past 0 to -1, then returns to 0. The graph of  $y = \sin x$  is shown below.



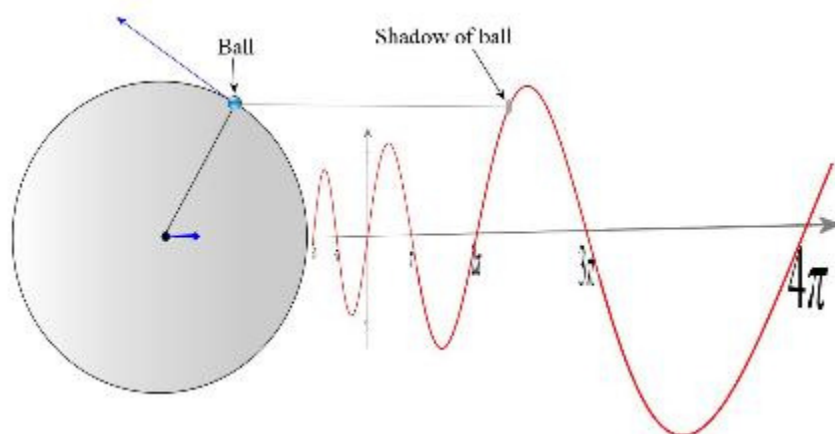
When we zoom in on a point, the graph looks more and more like a piece of a line. This line is called the **tangent line**. To find the derivative of  $\sin x$  at, say,  $x = z$ , we must find the slope of the tangent line at the point  $(z, \sin z)$ .

Our understanding of the derivative enables us to see that the derivative of  $\sin x$  is 0 for all values of  $x$  of the form  $\pi/2 + \pi k$ , where  $k$  is any integer, because these are the places where the sine function reaches its maximum and minimum values. (If the derivative weren't 0, then the graph wouldn't look like a piece of a horizontal line when we zoom in at these points. But that would mean that we could change  $x$  a little bit and find bigger and smaller values of sine nearby, contradicting the maximality or minimality at these points.)

But how can we find the derivative of  $\sin x$  at arbitrary points?

Let's build a contraption that creates the graph of the sine function. The contraption consists of a little ball travelling counterclockwise around a vertically-oriented unit circle at the constant speed of 1 unit per second.

We'll align the plane of the vertically-oriented circle so that it is perpendicular to a vertical wall. Now imagine projecting a shadow of the ball onto the wall using a light source that is extremely far away (so that the light rays are all parallel to each other and perpendicular to the wall).

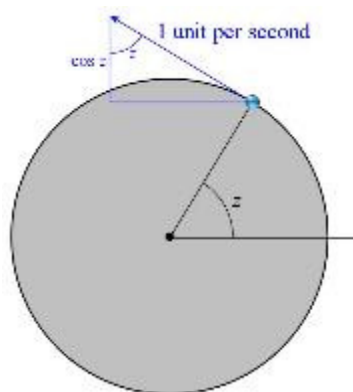


As the ball goes round and round the circle, its projected shadow on the wall will bob gently up and down. Because each unit around a unit circle corresponds to an angular change of exactly one radian, the angular speed of the ball is one radian per second. This means that the vertical height of the shadow is given by  $\sin t$ , where  $t$  is time measured in seconds. (Here, we are calibrating our distances so that the height on the wall corresponding to the projection of the center of the circle is considered to be 0.)

If we now move the circle in the direction perpendicular to the plane of the circle (and parallel to the wall) at a constant speed of 1 unit per second, the shadow of the ball will travel precisely along the graph of the sine function (see the figure above).

We would like to figure out the slope of the tangent line at  $(z, \sin z)$ . If, at the instant the shadow of the ball reached the point  $(z, \sin z)$ , we freed the ball from the circle and let the ball continue on with the exact same vertical and horizontal speed, the shadow of the ball would then leave the sine curve and start following the tangent at  $(z, \sin z)$ . After all, the tangent line at  $(z, \sin z)$  represents a line whose slope corresponds to the rate, in units per second, at which the shadow rises just at the instant the shadow reaches the point  $(z, \sin z)$ .

In other words, to determine the slope of the tangent line at  $(z, \sin z)$ , we must figure out the vertical speed in units per second right when the shadow reaches the point  $(z, \sin z)$ . The figure below left illustrates the exact moment when the shadow of the ball is over the point  $(z, \sin z)$ .



(The shadow of the ball is not depicted in this figure.) Because the ball is traveling around a circle, its velocity is perpendicular to the radial line to the ball. In this way, we can see that the vertical speed of the ball at this moment is  $\cos z$  units per second.

If, just at the moment when the shadow reaches  $(z, \sin z)$ , the ball is freed from the circle without changing its velocity, its shadow would, in one second, move 1 unit over in the direction of the positive horizontal axis of the graph and up by  $\cos z$  units. We conclude that the slope of the tangent line at  $(z, \sin z)$  is  $\cos z$ .

Thus, the derivative of  $\sin x$  with respect to  $x$  is  $\cos x$ .

## The derivative of the cosine function

The graph of cosine is a horizontal translation of the graph of sine:  $\cos x = \sin(x + \pi/2)$ .

Therefore, the derivative of  $\cos x$  is the derivative of  $\sin(x + \pi/2)$ , which we can compute by translating the derivative of sine. The derivative of  $\cos x$  with respect to  $x$  is  $\cos(x + \pi/2)$ , which is the same as  $-\sin x$ .

## The derivative of the tangent, secant, cosecant, and cotangent functions

The tangent function is  $\sin(x)/\cos(x)$ . Since we know the derivatives of  $\sin x$  and  $\cos x$ , we can apply the quotient rule to find the derivative of the tangent function. (The quotient rule was derived in the previous installment.) Can you fill in the details and show that the derivative of  $\tan x$  with respect to  $x$  is  $1/\cos^2(x)$ ?

Since  $\sec x = 1/\cos x$ ,  $\csc x = 1/\sin x$ , and  $\cot x = \cos(x)/\sin(x)$ , all of their derivatives can be computed using the general rules for differentiation that we have already discovered.

Here's a table of the derivatives of all six basic trigonometric functions:

$f(x)$	$\sin x$	$\cos x$	$\tan x$	$\sec x$	$\csc x$	$\cot x$
$f'(x)$	$\cos x$	$-\sin x$	$1/\cos^2(x)$	$\sec(x)\tan(x)$	$-\csc(x)\cot(x)$	$-1/\sin^2(x)$

## The derivative of inverse functions

Suppose  $f$  and  $g$  are inverse functions. Then  $f(g(x)) = x$ . If we differentiate with respect to  $x$  using the chain rule, we find that  $f'(g(x))g'(x) = 1$ . Hence  $g'(x) = 1/f'(g(x))$ . (Can you interpret this fact geometrically in terms of the graphs of  $f$  and  $g$ ?)

We can use this formula to find the derivative of an inverse function if we know the derivative of the function. To illustrate, let's compute the derivative of  $\arctan x$ :

Using the formula, we find that

$$\frac{d}{dx} \arctan x = \frac{1}{1/\cos^2(\arctan x)} = \cos^2(\arctan x).$$

Unravelling the trigonometric ratios, we find that  $\cos(\arctan x) = 1/\sqrt{1+x^2}$ . Hence,

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}.$$

For more problems on derivatives, see the Summer Fun problem set on page 27.



# Summer Fun!

The best way to learn math is to do math, so here are the 2015 Summer Fun problem sets.

We invite all members and subscribers to the Bulletin to send any questions and solutions to [girlsangle@gmail.com](mailto:girlsangle@gmail.com). We'll give you feedback and might put your solutions in the Bulletin!



The goal may be the lake, but who knows what wonders you'll discover along the way?

In the August issue, we will provide some solutions. You could wait until the August issue to see the answers, but you will learn a lot more if you first try to solve these problems on your own.

Some problems are quite a challenge and could take several weeks to solve, so please don't approach these problems with the idea that you must solve them all. Our main goal is to give you some interesting things to think about.

If you get stuck, try to formulate a related question that you can see a way to actively explore to get your mind moving and your pencil writing. If you don't understand a question, email us.

If you're used to solving problems fast, it can feel frustrating to work on problems that take weeks to solve. But there are things about the journey that are

enjoyable. It's like hiking up a mountain. Getting to the top rewards one with a spectacular view, but during the journey, there's a lot to see and experience. So here's a meta-problem for those of you who feel frustrated when doing these problems: see if you can dissolve that frustration and replace it with a relaxed, optimistic sense of adventure!

This is Summer Fun, not Summer Torture!

# Summer Fun!

# Telescoping Series

by Fan Wei

Here's a topic that I personally think is really fun.

Suppose we have a sequence of numbers  $a_1, a_2, a_3, a_4, \dots$ . The sequence may be finite or infinite. The associated series is the sum of these numbers  $a_1 + a_2 + a_3 + \dots$ . If the sequence is infinite, by the sum we mean the limit of the sequence  $s_k$  where

$$s_k = a_1 + a_2 + a_3 + \dots + a_k.$$

The sequence  $s_k$  is called the sequence of **partial sums** of  $a_k$ .

A **telescoping series** is a series  $a_1 + a_2 + a_3 + \dots$  where you can write  $a_k = b_k - b_{k+1}$  for some relatively nice sequence of numbers  $b_k, k = 1, 2, 3, \dots$

1. Let  $a_k$  and  $b_k$  be as explained above. Express  $s_n = a_1 + a_2 + a_3 + \dots + a_n$  in terms of the  $b_k$ .

(When you did Problem 1, did you find that terms cancelled metaphorically like the collapsing of a refracting telescope?) A consequence of Problem 1 is that if  $a_k$  is infinite, then the associated series is equal to the limit of  $b_1 - b_n$ , as  $n$  tends to infinity.

In Problems 2-5, we will consider the series

$$\frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \dots,$$

i.e. the series associated to the sequence defined by  $a_k = \frac{1}{1+2+3+\dots+k} = \frac{2}{k(k+1)}$ .

2. Show that  $a_k = \frac{2}{k} - \frac{2}{k+1}$ .

3. Show that the series is a telescoping series with  $a_k = b_k - b_{k+1}$  where  $b_k = \frac{2}{k}$ .

4. What's the simplest expression you can find for the  $n$ th partial sum of the series.

5. What is the sum of this infinite series?

For Problems 6-14, use the concept of telescoping series to compute the partial sums and, if the sum exists (i.e. the limit of the partial sums converges), determine the sum of the given infinite series  $a_1 + a_2 + a_3 + \dots$ . For the series considered in Problems 2-5, we gave  $b_k$ . But in the following problems you will have to figure out the sequence  $b_k$  such that  $a_k = b_k - b_{k+1}$ .



# Summer Fun!

6.  $\frac{3}{1 \cdot 2} - \frac{5}{2 \cdot 3} + \frac{7}{3 \cdot 4} - \frac{9}{4 \cdot 5} + \dots$  Here,  $a_k = (-1)^{k+1} \frac{2k+1}{k(k+1)}$ .

(Note the alternation of signs in the sum!)

7.  $\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \frac{1}{4 \cdot 5 \cdot 6} + \dots$

8.  $\frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{1}{3 \cdot 4 \cdot 5 \cdot 6} + \frac{1}{4 \cdot 5 \cdot 6 \cdot 7} + \dots$

9.  $\frac{1}{\sqrt{1} + \sqrt{2}} + \frac{1}{\sqrt{2} + \sqrt{3}} + \frac{1}{\sqrt{3} + \sqrt{4}} + \frac{1}{\sqrt{4} + \sqrt{5}} + \dots$

10.  $1 \times 1! + 2 \times 2! + 3 \times 3! + 4 \times 4! + \dots$

11.  $1^2 + 2^2 + 3^2 + 4^2 + \dots$

12.  $\sin 2 + \sin 4 + \sin 6 + \sin 8 + \dots$

(Hint: use the trigonometric identity  $2 \sin x \sin y = \cos(x - y) - \cos(x + y)$ .)

13.  $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \dots$

14.  $\arctan \frac{1}{2 \cdot 1^2} + \arctan \frac{1}{2 \cdot 2^2} + \arctan \frac{1}{2 \cdot 3^2} + \arctan \frac{1}{2 \cdot 4^2} + \dots$

15. Let  $a_k, k = 1, 2, 3, \dots$ , be a sequence. Suppose that we can find another sequence  $b_k$  such that

$$a_k = b_k - 2b_{k+1} + b_{k+2},$$

for  $k = 1, 2, 3, \dots$ . Express the partial sums  $s_k = a_1 + a_2 + a_3 + \dots + a_n$  in terms of the  $b_k$ . How does this relate to telescoping series?

17. (Generalization of Problems 7 and 8) Fix  $n$  to be a positive integer. Let

$$a_k = \frac{1}{k(k+1)(k+2) \cdots (k+n)}.$$

What is  $a_1 + a_2 + a_3 + \dots$ ?

Hint: Let  $b_k = 1/k$ . Show that  $a_k = \frac{1}{n!} \sum_{j=0}^n (-1)^j \binom{n}{j} b_{k+j}$ .



# Summer Fun!

# Induction

by Girls' Angle Staff

Master the technique of mathematical induction by solving these problems using induction (even if there are alternative proofs that do not use induction).

1. In this problem, we'll use induction to prove that the sum of the first  $n$  positive odd integers is equal to  $n^2$ .

A. Let's check the base case. Show that the first odd number is equal to  $1^2$ .

B. Now let's prove the inductive step. Fix some positive integer  $N$ . Assume that

$$1 + 3 + 5 + \dots + (2N - 1) = N^2.$$

Deduce that we must also have  $1 + 3 + 5 + \dots + (2N - 1) + (2N + 1) = (N + 1)^2$ .

Taken together, parts A and B imply that the sum of the first  $n$  positive odd integers is, indeed, equal to  $n^2$  for all positive integers  $n$ .

2. Make a table showing the sum of the first  $n$  positive perfect cubes for various  $n$ . Guess a formula for this sum. Prove your formula using induction.

3. Prove that  $(\sqrt{2} + 1)^n$  can always be written as  $a + b\sqrt{2}$ , where  $a$  and  $b$  are integers.

4. This is a classic application of induction. Instead of using induction to prove that a certain formula is true, we will use induction to prove that the number of elements in a certain family of sets is counted by the Fibonacci numbers. Let  $F_0 = F_1 = 1$  and define  $F_n$  recursively by declaring that  $F_{n+1} = F_n + F_{n-1}$  for all  $n > 0$ . The sequence  $F_k$  is known as the Fibonacci sequence.

For positive integers  $k$ , let  $N_k$  be the number of ways a 2 by  $k$  rectangle can be tiled with 2 by 1 dominos. The figure shows one way to tile a 2 by 7 rectangle with 2 by 1 dominos.



A. Think about an induction scheme that can be used to show that  $N_k = F_k$  for all positive integers  $k$ . What will the base case or cases be? What will the inductive step or steps be?

(**Spoiler Alert!**) Because of the way Fibonacci numbers are defined, an induction scheme well-suited to showing that some sequence corresponds to the Fibonacci sequence is to show, as base cases, that the first two terms in the sequence are consecutive Fibonacci numbers, and then prove that the sequence satisfies the same recursive formula as the Fibonacci numbers for the inductive step.

B. (base cases) Show that  $N_1 = F_1$  and  $N_2 = F_2$ .

C. (inductive step)

Show that  $N_{k+1} = N_k + N_{k-1}$  for  $k > 1$ .



# Summer Fun!

5. Let  $C_n$  denote the number of subsets of  $\{1, 2, 3, \dots, n\}$  that contain at least one even number. Prove (using induction) that  $C_n = 2^n - 2^{\lfloor (n+1)/2 \rfloor}$ , where  $\lfloor x \rfloor$  is the greatest integer less than or equal to  $x$ .

6. Let  $x_1, x_2, x_3, \dots, x_n$  be  $n$  positive numbers. The **arithmetic-geometric mean inequality** says that the arithmetic mean of these  $n$  numbers is always greater than or equal to their geometric mean, with equality if and only if all  $n$  numbers are equal. Let  $P_n$  be the statement:

$$\frac{x_1 + x_2 + x_3 + \dots + x_n}{n} \geq \sqrt[n]{x_1 x_2 x_3 \cdots x_n} \text{ with equality if and only if } x_1 = x_2 = x_3 = \dots = x_n.$$

We will use an induction scheme due to Cauchy to prove  $P_n$  for all integers  $n > 1$ . (For the sake of completeness, verify  $P_1$  separately.)

A. (base case) Show that  $P_2$  is true.

B. (inductive step 1) For  $n > 1$ , show that  $P_n$  implies  $P_{2n}$ . Hint: Split the  $2n$  numbers into two groups of  $n$  numbers each and use both the base case and  $P_n$  to deduce  $P_{2n}$ .

C. (inductive step 2) For  $n > 2$ , show that  $P_n$  implies  $P_{n-1}$ . Hint: If you have  $n - 1$  positive numbers and their arithmetic mean is  $a$ , apply  $P_n$  to the  $n$  numbers consisting of the original  $n - 1$  numbers together with  $a$ .

D. Explain how parts A, B, and C work together to prove  $P_n$  for all  $n > 1$ .

7. Define a sequence of polynomials  $T_n(x)$  recursively as follows. Let  $T_0(x) = 1$  and  $T_1(x) = x$ . For  $n > 1$ , define  $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$ . These polynomials are known as the **Chebyshev polynomials of the first kind**. (See the cover for a graph of  $T_4(x)$ .)

A. Prove by induction that the leading coefficient of  $T_n(x)$  is  $2^{n-1}$  for positive integers  $n$ .

B. Guess the constant term of  $T_n(x)$ . Prove your guess by induction.

C. Prove by induction that  $\cos nx = T_n(\cos x)$  for all nonnegative integers  $n$ . Hint: Use the trigonometric identity  $2 \cos(nx) \cos(x) = \cos((n+1)x) + \cos((n-1)x)$ .

D. Using Vieta's formulas, determine  $\prod_{k=0}^{n-1} \cos\left(\frac{2\pi k}{n}\right)$ .

8. Prove by induction that  $(\sqrt{2} + 1)^n$  can always be written as  $\sqrt{k+1} + \sqrt{k}$  for some positive integer  $k$ .



# Summer Fun!



# The Symmetric Group

by Noah Fechter-Pradines

Recall that a permutation of a set  $S$  is a one-to-one, onto map from  $S$  to itself. In the *Math Buffet* of the previous issue of this Bulletin, Cayley graphs of permutations (mostly of  $\{A, B, C, D\}$ ) are featured. In this Summer Fun problem set, we'll explore properties of  $S_n$  that underpin many of the phenomena we saw in making those Cayley graphs.

Let  $S_n$  be the set of all the permutations of  $\{1, 2, 3, \dots, n\}$ .

1. How many elements are there in  $S_n$ , in terms of  $n$ ? (This is referred to as the **order** of  $S_n$ . In general, the number of elements in any set  $S$  will be written  $|S|$ ). Write down all the permutations of  $S_4$  and check that the number you get is consistent with your previous answer.

We denote by  $1$  the permutation that fixes every element and call it the **identity** permutation.

In Problems 2-5, let  $p$ ,  $q$ , and  $r$  be permutations in  $S_n$ . Denote by  $pq$  the map that sends  $k$  to  $p(q(k))$ . In other words,  $pq$  is the composition of the maps  $p$  and  $q$ .

2. Show that  $pq$  is, itself, a permutation in  $S_n$ .

3. Show that  $p1 = 1p = p$ .

4. Show that  $(pq)r = p(qr)$ . That is, show that composition of permutations is **associative**. Because of associativity, we will write  $pqr$  for  $(pq)r$  or  $p(qr)$ .

5. Show that there is a unique permutation  $u$  in  $S_n$  such that  $pu = up = 1$ . The permutation  $u$  is called the **inverse** of  $p$  and is denoted  $p^{-1}$ .

Problems 2-5 show that  $S_n$  comes equipped with an associative binary operation (composition), that composing with  $1$  does not change a permutation, and that every permutation has an inverse. When we want to emphasize this additional structure on  $S_n$ , we call  $S_n$  a **group**. In fact,  $S_n$  is generally known as the **symmetric group** on  $n$  objects.

6. Use the existence of an inverse to show that for any element  $p$  in  $S_n$ , if  $p_1$  and  $p_2$  are distinct elements of  $S_n$ , then the elements  $pp_1$  and  $pp_2$  are also distinct. (What does this imply about Cayley graphs of  $S_n$ ?)

Let  $H$  be a nonempty subset of  $S_n$  with the property that it is **closed** under composition, that is, the composition of any two elements of  $H$  is also in  $H$ . Such a subset is called a **subgroup** of  $S_n$ . If  $H$  is not all of  $S_n$ , we say that  $H$  is a **proper** subgroup of  $S_n$ .

Note: In group theory, a subgroup is normally required to be a nonempty subset closed under both composition *and* inverses. Since  $|S_n|$  is finite, closure under inverses follows automatically from closure under composition.

Let  $S$  be a subset of  $S_n$ . The smallest subgroup  $H$  of  $S_n$  that contains  $S$  is called the subgroup **generated** by  $S$ . We also say that  $S$  **generates**  $H$ .

# Summer Fun!

7. Let  $H$  be a subgroup of  $S_n$ . If  $p$  is in  $H$ , show that  $p^{-1}$  is in  $H$ . Also, show that 1 is in  $H$ . Thus, every subgroup of  $S_n$  contains 1 and is closed under inverses of elements.

Before going further, let's introduce a way to represent permutations. Let  $p$  be a permutation of  $\{A, B, C, D\}$ . We can represent  $p$  by writing down the list  $p(A) p(B) p(C) p(D)$ , as done in the *Math Buffet* of the previous issue. This is known as **one-line** notation. Instead of one-line notation, we can represent a permutation with **cycle notation**. Let  $p$  be a permutation in  $S_n$ . In cycle notation, we pick a number, say  $x$ , and write it down. We then write  $p(x)$ , then  $p(p(x))$ , then  $p(p(p(x)))$ , and continue until just before we get back to  $x$ . We put parentheses around this list. If we've exhausted all the numbers 1 through  $n$ , we stop. Otherwise, we pick an unused number and do the same thing. For example, the permutation in  $S_5$  that sends 1 to 4, 2 to 2, 3 to 5, 4 to 3, and 5 to 1 is represented by  $(1435)(2)$  in cycle notation. In cycle notation, we usually omit singletons like the  $(2)$  in  $(1435)(2)$ .

8. Write the permutation in  $S_7$  represented by 6314725 in one-line notation, in cycle notation.

9. Let  $H$  be the subgroup of  $S_4$  generated by  $(123)$ . (Remember that we usually drop singletons from cycle notation, so in  $S_4$ ,  $(123)$  is the permutation  $(123)(4)$ , or, in one-line notation, 2314.) Write down all the elements in  $H$ . Is  $H$  a proper subgroup? What is  $|H|$ ?

10. Prove that for any  $n$ , the permutations  $(12), (23), (34), \dots, (n-1 \ n)$  generate  $S_n$ .

11. Prove that for any  $n$ , the permutations  $(12)$  and  $(123\dots n)$  generate  $S_n$ .

A permutation of the form  $(ab)$  where  $a$  and  $b$  are distinct is called a **transposition**.

Let  $A_n$  denote the subgroup of  $S_n$  generated by the set of elements that are products of 2 transpositions.

12. Show that  $A_n$  is a proper subgroup of  $S_n$ . **Can you determine  $|A_n|$ ?**

Let  $H$  be a subgroup of  $S_n$  and let  $p$  be in  $S_n$ . Define the **right coset**  $Hp$  to be the set of elements  $hp$  for all  $h$  in  $H$ . In general,  $Hp$  will not be a subgroup of  $S_n$ .

13. Consider the subgroup  $H$  of  $S_4$  generated by  $(12)(34)$  and  $(13)(24)$ . Write out the elements of  $H$ ,  $H(12)$ ,  $H(234)$ , and  $H(1324)$ . What do you notice?

14. Let  $H$  be a subgroup of  $S_n$  and let  $p$  be in  $S_n$ . Show that  $Hp = H$  if and only if  $p$  is in  $H$ .

**15. Let  $H$  be a subgroup of  $S_n$  and let  $p_1$  and  $p_2$  be in  $S_n$ . Suppose that  $Hp_1$  and  $Hp_2$  share an element. Show that  $Hp_1 = Hp_2$ . Show that every element of  $S_n$  is in some right coset  $Hp$ . Conclude that the right cosets of  $H$  partition  $S_n$ .**

Denote by  $[S_n : H]$  the number of distinct right cosets of  $H$ . We call  $[S_n : H]$  the **index** of  $H$  in  $S_n$ .

16. Use Problem 6 to show that every right coset of  $H$  contains  $|H|$  elements. From this and Problem 15, deduce the formula  $|S_n| = |H| [S_n : H]$ .



# Summer Fun!

This formula shows that the order of any subgroup of  $S_n$  must divide the order of  $S_n$ .

17. Define an analogous notion of **left coset** and redo problems 13-16 for left cosets instead of right cosets.

18. Look at the Cayley graphs in the *Math Buffet* of the previous issue of this Bulletin. Apart from the final, pentagonal graph, all of these are Cayley graphs of  $S_4$  (albeit thinking of  $S_4$  as permutations of  $\{A, B, C, D\}$  instead of  $\{1, 2, 3, 4\}$ ). For each, determine the elements of  $S_4$  used to define the edges of the graph and whether the subgroup that these elements generate is a proper subgroup or all of  $S_4$ . (Note: In those Cayley graphs, if an element  $g$  was used to define an edge, then the permutation  $p$  was connected to  $pg$  (not  $gp$ ).) Can you identify subgroups and their right cosets in the graphs?

So far, we have been thinking of  $S_n$  as an isolated object. However, a permutation in  $S_n$  is a map from the set  $\{1, 2, 3, \dots, n\}$  to itself. In general, if you have a set  $X$ , and, for each permutation  $p$  in  $S_n$  you are able to define a map  $m_p : X \rightarrow X$  in such a way that  $m_1$  acts as the identity map and  $m_p(m_q(x)) = m_{pq}(x)$  for all  $p$  and  $q$  in  $S_n$ , then you have created what is called an  **$S_n$ -action on  $X$** .

19. Let  $X = \{1, 2, 3, \dots, n\}$ . For  $p$  in  $S_n$ , define the map  $m_p : X \rightarrow X$  by setting  $m_p(x) = p(x)$ . Show that this defines an  $S_n$ -action on  $X$ .

20. Suppose you have an  $S_n$ -action on a set  $X$ . Let  $p$  be in  $S_n$  and suppose  $m_p(x) = y$ , where  $x$  and  $y$  are in  $X$ . Show that  $m_{p^{-1}}(y) = x$ .

From now on, we will denote the map  $m_p$  by  $p$  as well and use context to understand when  $p$  stands for a permutation or when it stand for a map from  $X$  to  $X$ .

Suppose you have an  $S_n$ -action on a set  $X$ . For any  $x$  in  $X$ , define the **stabilizer** of  $x$  to be the set of elements  $p$  in  $S_n$  such that  $p(x) = x$ . The stabilizer of  $x$  is denoted  $\text{Stab}(x)$ . The **orbit** of an element  $x$  in  $X$ , denoted  $\text{Orb}(x)$ , is the set of elements  $p(x)$  for all  $p$  in  $S_n$ .

21. Prove that  $\text{Stab}(x)$  is a subgroup of  $S_n$ . Prove that the number of elements in  $\text{Orb}(x)$  is equal to  $[S_n : \text{Stab}(x)]$ .

22. Let  $X_k$  be the set of subsets of  $\{1, 2, 3, \dots, n\}$  of size  $k$ . For each  $p$  in  $S_n$ , define a map from  $X_k$  to itself by declaring that  $p(T) = \{p(t) \mid t \in T\}$ , where  $T$  is an element of  $X_k$ .

A. Show that this defines an  $S_n$ -action on  $X_k$ .

B. For any  $T$  in  $X_k$ , prove that  $|\text{Stab } T| = k!(n - k)!$ .

C. Show that  $\text{Orb}(T) = X_k$ .

D. Conclude that the number of subsets of  $\{1, 2, 3, \dots, n\}$  of size  $k$  is equal to  $\binom{n}{k} = \frac{n!}{k!(n - k)!}$ .

# Summer Fun!

# Derivatives

by Ken Fan

These problems can be considered as part of our ongoing series on the derivative. Unless otherwise noted, all functions are assumed to map the real numbers to itself and have derivatives of all orders. By “derivatives of all orders,” we mean that you can repeatedly differentiate a function over and over. Denote by  $f^{(n)}(x)$  the  $n$ th derivative of  $f(x)$  with respect to  $x$ , that is,  $f^{(0)}(x) = f(x)$  and  $f^{(n+1)}(x) = df^{(n)}(x)/dx$  for  $n \geq 0$ .

1. (Leibniz’s rule) The product rule says that the derivative of  $f(x)g(x)$ , with respect to  $x$ , is  $f'(x)g(x) + f(x)g'(x)$ . Express the  $n$ th derivative of  $f(x)g(x)$  with respect to  $x$  in terms of higher order derivatives of  $f$  and  $g$ .

2. Intuitively, we explained that if  $f$  is differentiable at  $x = z$ , then when you zoom into the graph around  $(z, f(z))$ , the graph looks more and more like part of a straight line, and we described the derivative of  $f(x)$  at the point  $x = z$  to be the slope of that line. Note that the second derivative of a straight line is exactly zero. Consider the function  $f(x) = x^2$ . Note that  $f'(x) = 2x$  and  $f^{(2)}(x) = 2$ , not 0. Explain why the graph of  $f(x)$  “looks locally like part of a straight line,” yet has a nonzero second derivative.

3. Let  $p(x)$  be a polynomial of degree  $d$  with distinct roots and lead coefficient 1. Show that the sum of the roots of  $p$  is equal to  $-f^{(d-1)}(0)/(d-1)!$ .

4. Let  $p(x)$  be a polynomial. Show that  $p(x)$  has multiple roots if and only if  $p(x)$  and  $p'(x)$  share a non-constant common factor.

5. Here is a rigorous definition of the derivative. We say that  $m$  is the derivative of  $f(x)$  with respect to  $x$  at  $x = z$  if and only if for all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that

$$|f(z+h) - (mh + f(z))| < \varepsilon|h|$$

for all  $0 < |h| < \delta$ . Can you derive all the properties of the derivative we have discussed in our series on the derivative starting from this definition?

6. If  $f'(x) = g'(x)$  for all  $x$ , must it be true that  $f(x) = g(x)$ ? If so, explain why. If not, provide a counterexample.

7. Deduce the double-angle formula for cosine by taking the derivative of both sides of the double-angle formula for sine:  $\sin(2x) = 2\sin(x)\cos(x)$ . Deduce the half-angle formula for cosine by differentiating both sides of the half-angle formula for sine:  $\sin(x/2) = \sqrt{(1 - \cos x)/2}$ .

8. Let  $f(x) = \arctan x$ . Determine  $f^{(n)}(0)$  for all nonnegative integers  $n$ .

9. Find a real-valued function  $f$  defined on nonnegative real numbers such that  $f'(x) = 1/f(x)$  for all  $x \geq 0$ .



# Summer Fun!

# Notes from the Club

These notes cover some of what happened at Girls' Angle meets. They are far from being complete. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

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Session 16 – Meet 12    Mentors:    Bridget Bassi, Karia Dibert, Jennifer Matthews,  
May 7, 2015                                      Wangui Mbuguiro, Jane Wang, Sibo Wang

We held our traditional end-of-session Math Collaboration!

Members arrived to find a big locked-up treasure chest on a table. Next to the chest sat a pamphlet containing a batch of math problems that covered material we worked on this spring along with instructions for how to find the combination to the lock.

About halfway through, a group of girls succeeded in deciphering a long secret message. The last sentence of the message read, “The combination to the big treasure chest lock is up left down right down.” At this moment, staff thought they’d hurry over to the chest and try the combination. However, after conferring among themselves, they decided that it couldn’t be the combination because there were too many other math problems whose answers weren’t needed to decode the secret message. Surely, they thought, in a well-constructed puzzle, there wouldn’t be so many wasted problems! So instead of trying the combination, they forged ahead with the solving of the remaining unsolved math problems.

Half an hour later, when nearly all the math problems had been solved, a member insisted that they try the combination. With not much else left to do, they gathered around the big treasure chest, most still dubious that the combination would work. They entered the combination, and click, the lock opened! They stood around looking puzzled as though they were collectively wondering, “why all the other math problems?” They didn’t even express enthusiasm for opening the big treasure chest because they partly felt it was a bit of a letdown to open the chest in this manner.

Still, one of them lifted the lid – and discovered a smaller treasure chest tucked inside! There was also another math problem that required the solutions to all the other math problems to solve in order to get the combination of the lock on the smaller chest.

Here are a few of the problems from our end-of-session Math Collaboration:

Pauline made a 45-pointed star like this: First, she carefully plotted the 45 vertices of a regular 45-gon. She then connected each vertex to each of the two vertices 22 vertices away (i.e. the two vertices farthest away). The result was a 45-pointed star with 45 identical tips. What is the angle measure of one of these tips in degrees?

Let  $A$  be the largest number you can express using 16 ones, addition, multiplication, and parentheses. Let  $B$  be the largest number you can express using 8 ones, addition, multiplication, and parentheses. What is  $A/B$ ?

In a Cayley graph for the permutations on 10 letters, what would be the longest possible monochromatic loop?

Let  $\omega$  be the first countable ordinal. What whole number  $n$  makes the equation  $\omega^n = \sum_{x < \omega^3} x$  true?



# Calendar

## Session 16: (all dates in 2015)

January	29	Start of the sixteenth session!
February	5	
	12	
	19	No meet
	26	
March	5	
	12	
	19	
	26	No meet
April	2	
	9	Bathsheba Grossman, Sculptor
	16	
	30	
May	7	

## Session 17: (all dates in 2015)

September	17	Start of the seventeenth session!
	24	
October	1	
	8	
	15	
	22	
	29	
November	5	
	12	
	19	
	26	Thanksgiving - No meet
December	3	
	10	

Girls' Angle has been hosting Math Collaborations at schools and libraries. Math Collaborations are fun math events that can be adapted to a variety of group sizes and skill levels. For more information and testimonials, please visit [www.girlsangle.org/page/math\\_collaborations.html](http://www.girlsangle.org/page/math_collaborations.html).

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: [girlsangle@gmail.com](mailto:girlsangle@gmail.com).

Good luck to NASA as New Horizons makes its closest approach to Pluto on July 14, 2015. New Horizons has been in flight since January 19, 2006. That's older than our youngest members!

# Girls' Angle: A Math Club for Girls

## Membership Application

**Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.**

Applicant's Name: (last) \_\_\_\_\_ (first) \_\_\_\_\_

Parents/Guardians: \_\_\_\_\_

Address (the Bulletin will be sent to this address):

Email:

Home Phone: \_\_\_\_\_ Cell Phone: \_\_\_\_\_

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

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The \$36 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- ☐ Enclosed is a check for \$36 for a 1-year Girls' Angle Membership.
- ☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to [girlsangle@gmail.com](mailto:girlsangle@gmail.com).



**A Math Club for Girls**

# Girls' Angle Club Enrollment

**Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!**

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

**Who are the Girls' Angle mentors?** Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

**What is the Girls' Angle Support Network?** The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

**What is Community Outreach?** Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

**Who can join?** Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

**How do I enroll?** You can enroll by filling out and returning the Club Enrollment form.

**How do I pay?** The cost is \$20/meet for members and \$30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

**Where is Girls' Angle located?** Girls' Angle is located about 12 minutes walk from Central Square on Magazine Street in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

**When are the club hours?** Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at [www.girlsangle.org/page/calendar.html](http://www.girlsangle.org/page/calendar.html) or send us email.

**Can you describe what the activities at the club will be like?** Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

**Are donations to Girls' Angle tax deductible?** Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

**Who is the Girls' Angle director?** Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

**Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities?** Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, executive director of Science Club for Girls  
Yaim Cooper, lecturer, Harvard University  
Julia Elisenda Grigsby, assistant professor of mathematics, Boston College  
Kay Kirkpatrick, assistant professor of mathematics, University of Illinois at Urbana-Champaign  
Grace Lyo, Instructional Designer, Stanford University  
Lauren McGough, graduate student in physics, Princeton University  
Mia Minnes, SEW assistant professor of mathematics, UC San Diego  
Beth O'Sullivan, co-founder of Science Club for Girls.  
Elissa Ozanne, associate professor, The Dartmouth Institute  
Kathy Paur, Kiva Systems  
Bjorn Poonen, professor of mathematics, MIT  
Gigliola Staffilani, professor of mathematics, MIT  
Bianca Viray, assistant professor, University of Washington  
Karen Willcox, professor of aeronautics and astronautics, MIT  
Lauren Williams, associate professor of mathematics, UC Berkeley

**At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics?** We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

# Girls' Angle: Club Enrollment Form

Applicant's Name: (last) \_\_\_\_\_ (first) \_\_\_\_\_

Parents/Guardians: \_\_\_\_\_

Address: \_\_\_\_\_ Zip Code: \_\_\_\_\_

Home Phone: \_\_\_\_\_ Cell Phone: \_\_\_\_\_ Email: \_\_\_\_\_

Please fill out the information in this box.

**Emergency contact name and number:** \_\_\_\_\_

**Pick Up Info:** For safety reasons, only the following people will be allowed to pick up your daughter. Names:

**Medical Information:** Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

**Photography Release:** Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

**Eligibility:** Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

**Personal Statement (optional, but strongly encouraged!):** We encourage the participant to fill out the optional personal statement on the next page.

**Permission:** I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

\_\_\_\_\_  
(Parent/Guardian Signature) Date: \_\_\_\_\_

Participant Signature: \_\_\_\_\_

Members: Please choose one.

- ☐ Enclosed is \$216 for one session (12 meets)
- ☐ I will pay on a per meet basis at \$20/meet.

Nonmembers: Please choose one.

- ☐ I will pay on a per meet basis at \$30/meet.
- ☐ I'm including \$36 to become a member, and I have selected an item from the left.

☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to [girlsangle@gmail.com](mailto:girlsangle@gmail.com). Also, please sign and return the Liability Waiver or bring it with you to the first meet.



**Personal Statement (optional, but strongly encouraged!):** This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

### Girls' Angle: A Math Club for Girls Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

\_\_\_\_\_,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: \_\_\_\_\_ Date: \_\_\_\_\_

Print name of applicant/parent: \_\_\_\_\_

Print name(s) of child(ren) in program: \_\_\_\_\_